MATHEMATICS APPLIED TO ELECTRICAL ENGINEERING

A SERIES OF MONOGRAPHS
ON
ELECTRICAL ENGINEERING

Under the Editorship of

H. P. YOUNG, M.I.E.E., M.A.I.E.E.

Head of the Electrical Power and Machinery Section, The Polytechnic, London

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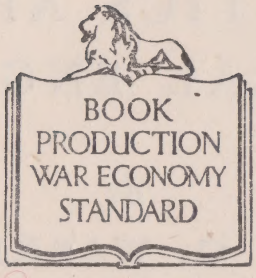

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EDITORIAL PREFACE

THE advances made within the realm of electrical engineering during the twentieth century have been phenomenal.

In the field of electric power supply the National Electricity Scheme has been conceived, designed and put into commission, thus increasing the reliability and availability of the service. Meanwhile the allied sciences of electrical communications and high frequency technique have moved forward with gigantic steps : indeed, it is hardly too much to say that, due to the advent of the electronic tube and its application in both the power and communications branches, certain aspects of electrical engineering practice have been completely revolutionized.

In view of these epoch-making advances, it is not surprising that the electrical engineer whose college days are some years behind him finds it necessary to do a considerable amount of reading in order to keep abreast of modern developments, while the student who is reading for a University degree is nowadays expected to possess at least an outline of the field of electrical physics in addition to a comprehensive knowledge of electrical technology.

While, however, the literature dealing with these advances is both comprehensive and voluminous, it is also scattered and thus available only by extensive literary research. The aim of each monograph published in this series is to give a modern orientation of a particular subject within the confines of a small book, thus obviating the necessity for searching through the transactions of innumerable learned societies. At the same time, for those wishing further to extend their knowledge, each monograph contains references to the more important publications relating thereto.

The satisfactory accomplishment of this object necessarily implies that the author must fulfil the dual function of collator

and interpreter. For this reason, one of the editor's most important tasks has been to induce acknowledged authorities to write for this series. In this respect, he has been fortunate beyond his expectation, and each monograph has been written by an author who is eminent in his chosen field and thus writes with the authority which is the result of intimate knowledge.

It is hoped that the Monographs will succeed in filling a gap which has hitherto existed in scientific literature.

H. P. YOUNG.

FOREWORD

THIS book deals with the applications of mathematical methods to Engineering Science. These applications are continually extending their usefulness and as the computations in connection with them are often tedious and laborious, the author has done well in showing how the necessary calculations can sometimes be abbreviated by applying both graphical and numerical methods of computation.

To two classes of readers this book is specially helpful. First to those who have studied the elementary theory of the calculus and of differential equations at school or college and desire to refresh their memory so that they can apply their knowledge to invention and research. Secondly there are those engaged in professional engineering work who discover new solutions which generally appear in the form of infinite series. They have to be careful that these series are convergent and can be computed approximately without excessive labour. For this purpose the instruction given in this book on harmonic analysis, conjugate functions, Heaviside's operational methods, Bessel's functions, etc., can be easily understood and is most helpful. The great accuracy now required in many practical problems connected with the transmission of electrical power, of telephonic signals, of television, etc., used in everyday research work practically prohibits the computation of the necessary curves from the older formulæ owing to the labour and time involved.

The many examples worked out by the author have been happily chosen and his solutions are obtained very neatly. The type is easy to read and the diagrams are constructed

with the minimum number of lines and are very lucid. We can warmly recommend this book to electrical engineers with a mathematical bent. Those engaged in laboratory research work will also find it very helpful.

ALEXANDER RUSSELL.

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AUTHOR'S PREFACE

In few branches of applied science is a comprehensive knowledge of mathematical methods of greater value than in electrical engineering. Yet it is only within comparatively recent years that the mathematical needs ot the engineer have been specifically recognized. For the engineer who has already a fair mathematical equipment several books, both of a specialist and of a general nature, have been written. With these excellent works the author makes no attempt to compete. It must be recognized, however, that the majority of problems arising in electrical practice and research call for a ready facility in the application of mathematics of a comparatively modest standard. It is mathematics of this nature with which the author is chiefly concerned. The outlook of the mathematician differs from that of the engineer. While the latter is not blind to the beauty of many of the mathematical methods employed, his interest must necessarily centre in the practical problem to which the mathematics is applied. As an engineer he must judge mathematics by its applicability. This is the problem which the author has faced in selecting material for the present book. It is hoped that the selection which has been made will be of value both to practical engineers and to students of electrical engineering. In making a choice the author has been guided by his own needs in some thirty years' experience of engineering education and research.

It will be recognized that several chapters treat of special mathematical methods and problems and may be omitted on a first reading if they possess no immediate interest. Again, these chapters are intended to serve as a sufficient exposition of the subjects with which they deal to be applicable to many practical problems. They may be read with or without reference to the earlier text.

Chapter V is based upon a collection of material to which

the author has found he has had to make constant reference. It is hoped that it will be found useful to many readers.

The author gratefully acknowledges the help which he has received from many books (some of which are named in the Bibliography) and numerous discussions with friends and colleagues. Particularly he is indebted to Mr. H. P. Young, M.I.E.E., for his interest and assistance during the writing of the book, and to Mr. D. E. Barnes, B.Sc., A.M.I.E.E., who read the whole of the manuscript and made many useful suggestions and corrected a number of errors.

Permission of the Ordnance Committee to publish the book is gratefully acknowledged.

A. G. W.

IGHTHAM.
July 1939.

CONTENTS

PAGE

EDITORIAL PREFACE v

FOREWORD BY ALEXANDER RUSSELL vii

AUTHOR'S PREFACE ix

CHAPTER I

NUMBERS—REAL AND COMPLEX 1

Real numbers—The complex number—Argand diagram—Multiplication of complex numbers—Arithmetical interpretation of j—The exponential theorem—The logarithmic expansion—The complex number expressed as an exponential function—The hyperbolic functions—Examples.

CHAPTER II

PRINCIPLES OF DIFFERENTIATION AND INTEGRATION. 12

Introduction—Differentiation (observer A)—Integration (observer B)—Differentiation and integration; Reciprocal relations.

CHAPTER III

METHODS AND RESULTS OF DIFFERENTIATION . . 18

Added constant—Constant multiplier—Sum or difference—Product of functions—Quotient—Function of a function—Inverse function—Logarithmic differentiation—Differentiation of standard forms—Examples—Successive differentiation—Successive differentiation of a product; Leibnitz's theorem—Maclaurin's theorem—Taylor's theorem—Differentials—Maxima and minima.

CHAPTER IV

METHODS AND RESULTS OF INTEGRATION . . . 30

Introduction—Constant of integration—General laws of integration—General processes of integration—Method of substitution—Definite integrals by the method of substitution—Convenient substitutions—Integration by parts—Examples.

CHAPTER V

A BRIEF SURVEY OF FUNDAMENTAL ELECTROSTATIC, MAGNETIC, AND ELECTROMAGNETIC RELATIONS . 40

Electrostatics—Forces between electric charges—Electric field; lines of force—Electric potential—Electric potential due to a point charge—Equipotential surfaces—Gauss's theorem—Digression on partial differentiation—Poisson's equation—Laplace's equation—

PAGE

Tubes of force—Surface density of charge—Equality of magnitude of charges at the ends of a tube—Electric displacement—Capacitance—Replacement of an equipotential surface by a conducting surface—Capacitance of a sphere—Capacitance of two concentric spheres—Capacitance of parallel plate condenser—Potential energy of a condenser—Energy of the electric field.

Magnetism—Force between two magnetic poles; magnetic moment of a magnet—Magnetic field; lines of force—Magnetic potential—Magnetic potential due to a short bar magnet—Potential due to a magnetic shell—Intensity of magnetization—Magnetic field between two equally and uniformly magnetized poles.

Electrodynamics—Magnetic field produced by an electric current—Force upon a conductor in a magnetic field—Magnetic field due to any element of a circuit—Magnetic moment of an electric circuit—Magnetic potential due to an electric current in a closed circuit—Force on a moving charge in a magnetic field—Electromotive force generated in a conductor moving in a magnetic field—Work done in changing the magnetic flux linking a circuit—Inductance—Energy associated with a current in an inductance—Vector fields; potential—Line integral in the electric field—Potential in a magnetic field—Curl of a vector—Magnetization of iron—Units; laws of electrodynamics—Numerical relations between the units.

CHAPTER VI

ELECTRICAL PROBLEMS 74

Average value of a sinoidal function—R.M.S. value of a sinoidal function—Intensity of magnetic field near to a straight wire carrying a current—Magnetic field due to a circular current—Magnetic field due to a circular current at a point in its plane—Intensity of the magnetic field at the centre of a square—Forces between conductors conveying currents—Force between two indefinitely long parallel conductors—Pinch effect—Calculation of self inductance of a circuit—Inductance of a coaxial cable—Inductance of a pair of parallel wires—Capacitance of a coaxial cable—Capacitance of a pair of parallel wires (approximate solution)—Capacitance of a pair of parallel wires (exact solution)—Dielectric stress in a coaxial cable; optimum size of central conductor—Optimum dimensions for a high frequency coaxial feeder—Pull of a solenoid upon an iron plunger; moving coil ammeter—Work done in increasing the area of an electric circuit—Movement of electrons in an electric field—Electric deflection of a cathode ray—Magnetic deflection of a cathode ray.

CHAPTER VII

HARMONIC AND VECTOR QUANTITIES; SYMMETRICAL COMPONENTS 96

Introduction—Harmonic quantities, vectors and rotating vectors—Vectorial representation of a wave motion—Application to alternating currents—Vector admittance—Inversion and vector loci—Inversion of a straight line—Inversion of a circle—Resonant circuit—Coupled circuits—Power in an alternating current circuit—Polyphase systems—Balanced and unbalanced three-phase systems—Power in a three-phase balanced circuit—Symmetrical components—Derived systems—Power in an unbalanced three-phase system.

CHAPTER VIII

PAGE

PARTIAL DIFFERENTIATION 120

Partial differentiation—Total differential coefficient—Maxima and minima.

CHAPTER IX

FURTHER METHODS OF INTEGRATION 125

$\int dx/X\sqrt{Y}$—Reduction formulæ—The definite integrals $\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta$ and $\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta$, n being integral.—Investigation of $\int \sin^m\theta \cos^n\theta\, d\theta$—A note upon the gamma function $\Gamma(p)$—The definite integrals expressed in terms of gamma functions—Double integration—Determination of an area from the polar equation—Triple integration—Differentiation under an integral sign—The beta function—Another form of the beta function—The gamma function—Relation between beta and gamma functions—Values of particular integrals in terms of gamma functions—Integral values of the gamma functions.

CHAPTER X

SIMPLE DIFFERENTIAL EQUATIONS AND EXAMPLES . 139

Introduction—Number of constants of integration—Equations of the first order and of the first degree—Current in an inductive circuit—Current in a capacitative circuit—Time constant.

CHAPTER XI

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS 151

Introduction—The equation with constant coefficients; second member zero—Equal roots in the auxiliary equation—Complex roots in the auxiliary equation—The operator D—The operator D (*continued*)—The polynomial in D—The particular integral—Special methods of obtaining the particular integral

CHAPTER XII

MORE GENERAL OPERATIONAL METHODS . . . 164

Introduction—Definitions and operators—The operator D and rules relating to it—Expansion of inverse operators—$F(D).\epsilon^{\phi(x)}$; Shifting—The circular and hyperbolic functions—The linear differential equation of the first order—The linear differential equation with constant coefficients—Operations upon zero; the complementary function—Inverse operations upon unity.

CHAPTER XIII

EXAMPLES INVOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS . . . 177

Series circuit containing resistance, inductance, and capacitance—Synchronous spark transmitter—The damped vibration considered generally—Moving-coil galvanometers—The ballistic galvanometer.

CHAPTER XIV

PAGE

SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS; PRACTICAL EXAMPLES 194

Introduction—The transformer—The single-phase alternator—Transformer with capacitative circuit—The valve oscillator—The moving-coil speaker—The telephone diaphragm.

CHAPTER XV

DIFFERENTIAL EQUATIONS (*continued*) 214

The homogeneous linear differential equation—Exact linear differential equations—Factorisation of the operator—Equation in commutative operators—Equations which do not contain y directly—Equations of the second order—Complete solution when an integral is known—Change of the dependent variable.

CHAPTER XVI

THE SOLUTION OF DIFFERENTIAL EQUATIONS IN SERIES 226

Method; the indicial equation—Values of m differing by an integer—Legendre's equation.

CHAPTER XVII

BESSEL FUNCTIONS 231

Definitions—Bessel's equation of zero order—Bessel's equation of order n—The second solution to the equation of order n, when n is integral—The modified Bessel functions $I_n(x)$ and $K_n(x)$—Integral forms of Bessel functions—Definite integrals for $J_0(x)$ and $I_0(x)$.

CHAPTER XVIII

PROBLEMS INVOLVING BESSEL FUNCTIONS, ETC. . 243

The grid rectifier—High frequency current in a cylindrical conductor—Ber, bei, ker, and kei functions—Solid wire—Conditions at very high frequencies—The hollow conductor—Distribution of current in the solid conductor—Eddy current loss in the core of a solenoid—Conditions at very high frequencies—Resistance of wide parallel strip—Resistance of strip at power frequencies—High-frequency resistance of strip—Penetration of flux into a lamination at high frequencies.

CHAPTER XIX

SOME PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATION 265

Some commonly occurring equations—Electromagnetic radiation—Particular solution: plane wave—Particular solution: spherical wave—Transmission along wires—Steady sinoidal conditions—Infinitely long line—Distortionless line—Line with small losses—Impedance of finite terminated line—The equation $\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}$—The telegraph equation—Arrival curves.

CHAPTER XX

PAGE

FOURIER'S SERIES AND HARMONIC ANALYSIS . . 286

Introduction—Fourier's half-range series—Harmonic curves—The trapezoidal wave—Rotating wave of flux in the air gap of a three-phase machine—Rectified sine wave—Harmonic analysis : direct method—Other methods—The ninth harmonic.

CHAPTER XXI

HEAVISIDE'S OPERATIONAL CALCULUS 306

Introduction—The unit function—The expansion theorem—Duhamel's integral—The line equations—The telegraph equation—Open circuited transmission line having negligible losses—Transmission line having resistance, inductance, capacitance and leakance.

CHAPTER XXII

CONJUGATE FUNCTIONS 328

Introduction—Laplace's equation in two dimensions—Conformal representation—The transformation $W = A \log z$—Ratio of transformation—The Schwarz-Christoffel transformation—Successive transformations—Distribution of current and potential in a conductor of uniform thickness whose breadth suddenly changes—Increase of resistance due to change of section—Flow of current from a point source into a sheet of resistive material—Flow of current from a point source into a sheet of resistive material having different conductivities along the two axes.

APPENDIX I 361

Circular, Hyperbolic and Exponential Functions, Differentiation, Integration, etc.

APPENDIX II 374

Electrical and Magnetic Equations

APPENDIX III 377

A Note on Dimensions

BIBLIOGRAPHY 379

INDEX 380

CHAPTER I

NUMBERS—REAL AND COMPLEX

Real Numbers.

The elementary operations of mathematics are addition, subtraction, multiplication and division. The processes may be associated with things or quantities. The rules applying to each differ.

The addition of two sheep to three sheep results in a new unity—five sheep ; the addition of two goats to three goats gives a sum of five goats. A step of importance was made when the parallelism of these two cases was recognised.* The sum of two sheep and three sheep could then be determined by one who possessed no sheep. In brief, the validity of the operation of addition is independent of the nature of the subject upon which the operation is performed. And so one passes from the consideration of the things themselves to the numbers or operators.

This abstraction is not complete, for it is subject to the limitations of *things*. The principal of these limitations is that only the positive integers have significance. A negative sheep is meaningless and half a sheep is not sheep, but mutton. It is true that in particular cases a useful significance may be attached to portions (e.g. half an apple), but that significance partakes of the nature of quantity and passes beyond the region of things. With this limitation one may proceed with the abstracted operators.

Two added to three is always five ; the same unity may equally well be the sum of one and four. The result gives no clue to the operation or the numbers upon which it has been performed. But immediately a further limitation of the abstraction is disclosed, the numbers must be associated with

* The first man who noticed the analogy between a group of seven fishes and a group of seven days made a notable advance in the history of thought. (Whitehead).

the same kind of thing. The addition of two sheep to three goats gives an assemblage but not a unity and the result can be expressed as nothing more, or less, than two sheep plus three goats. The limitation may be revealing. If the animal contents of two fields are identical and one contains two sheep and three goats, then the other must also contain two sheep and three goats. The point may appear trivial, but it is fundamental; it leads later to important results.

Still confining attention to things, subtraction is subject to the further limitation that it is impossible to take a greater number from a less.

From an analytical point of view multiplication is only repeated addition and can always be performed.

Division is subject to the limitation that only in certain cases can it be exactly performed. Two into seven goes three and one over is the only answer if one knows nothing of the nature of the subject. If seven cows are to be equally divided between two farmers they can have only three each. If seven children are each to have half an apple, four apples must be cut. With quantities and measures some of the processes become simpler and more definite. Fractions have as much significance as integers; the nature of a yard is not altered by division. If two women share seven yards of silk they have three and a half yards each.

When a yard is measured to the right, a negative yard can have a meaning, namely, a yard to the left. But when directions are not so severely restricted but may be left or right, up or down, or forwards or backwards, the use of positive and negative is inadequate for description. The negative quantity of the algebraist is often a convenient fiction, stranger than truth. A negative pint is inconceivable. Income tax cannot be collected by the issue of negative pound notes. The signs + and − are operators, not adjectives, and they operate upon positive quantities.

Multiplication considered as repeated addition presents no difficulties. The multiplier then expresses the number of times the multiplicand is added. Strictly speaking, this is the only true form of multiplication, though the process is often extended to the multiplication of quantities. Such a process has no inherent meaning; when a meaning exists it derives from a

definition. The artificiality of the process is indicated by the following two examples:

2 feet $\times$ 3 pounds = 6 foot pounds (an amount of work), the multiplication of two collinear vectors giving a scalar product.

3 pounds $\times$ 2 feet = 6 pound feet (a torque), the multiplication of two orthogonal vectors giving a vector product.

Many other products of quantities have useful meanings attached. These, however, need not concern us from an analytical point of view. Abstracting the numbers from the quantities, the former example above can be considered as merely the product of 2×3. Whether the unit product 1 foot $\times$ 1 pound has any useful physical interpretation is, for the moment, beside the point. Mathematically we are only concerned with the numbers, for which we have to discover the properties and rules of operation.

The Complex Number.

If from an origin O on a straight line OX a fixed segment OP_1 be taken (P_1 being to the right of O), a length OP_x can be measured such that the ratio OP_x/OP_1 is equal to any real number (Fig. 1). If x is positive P_x is to the right of O, if

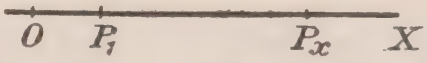


Fig. 1.

negative to the left. It should be noted that the abstracted number x is not represented by the length OP_x but by its ratio to the unit length OP_1. (Economy of language may in some instances justify referring to x as being represented by OP_x so long as the actual condition is recognised.)

Conversely, any displacement of the point P_x upon the line OX may be represented by a real number x. If, however, P_z is any point in a plane two real numbers are required for its specification, conveniently x its displacement parallel to OX and y its displacement perpendicular to OX (Fig. 2). The position of P_z may be defined, by analogy, as $OP_z/OP_1 = z$. z is a *complex* multiplier which operating

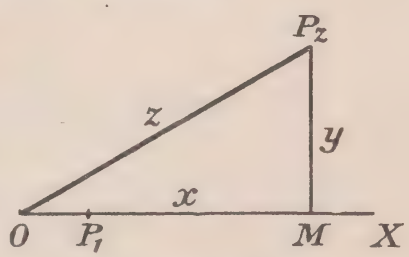


Fig. 2.

upon OP_1 gives OP_z, just as the *real* multiplier x operating upon OP_1 gives OP_x. Now the displacement OP_z may be considered as a displacement $OM = x.OP_1$ together with a displacement MP_z of *magnitude* $y.OP_1$ but at right angles to OX. This is expressed by writing $MP_z = jy.OP_1$. j is an operator (termed imaginary) which, acting upon a real length, turns it through a right angle (or advances its phase by $\pi/2$). We thus arrive at the *complex* number $z = x + jy$.

To every point in the plane there corresponds one, and only one, complex number. To any complex number there corresponds one, and only one, point in the plane. x and y are termed respectively the *real* and *imaginary* parts of z and are frequently written $x = R(z)$ and $y = I(z)$.

The following relations are obviously true.

If $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$

then $$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \quad . \quad . \quad . \quad (1)$$

and $$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \quad . \quad . \quad . \quad (2)$$

Or again if $z_1 = x_1 + jy_1 = z_2 = x_2 + jy_2$

then $$x_1 = x_2 \text{ and } y_1 = y_2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This relation (3) is of importance; if two complex quantities are equal then their real and imaginary parts are separately equal.

Argand Diagram.

The position of the point P_z may also be represented by its distance from the origin $= r.OP_1$ where $r = \sqrt{x^2 + y^2}$, and θ, the angle which the radius vector makes with the axis of x (Fig. 3). (This diagram was published by Argand in 1806 and is usually associated with his name.) Then $r \cos\theta = x$ and $r \sin\theta = y$ and

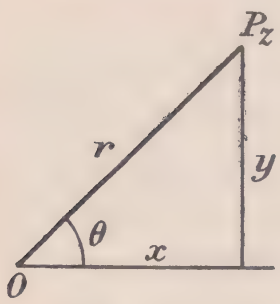


FIG. 3.

$$z = x + jy = r(\cos\theta + j \sin\theta) \quad . \quad . \quad (4)$$

r is the numerical magnitude of z and is called the *modulus* of z, written $|z|$ or mod z. The angle θ is called the *argument*, or *amplitude*, or *phase*, of z, written $\theta = \arg z$.

The effect of multiplying any length by z is therefore to increase its magnitude in the ratio r or $|z|$, at the same time turning it through a positive angle θ. This is expressed by the equation

$$z = |z|(\cos\theta + j \sin\theta) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is seen that although the modulus of a complex number is unique, its argument is not. The substitution of $2n\pi + \theta$ for θ, where n is any integer, does not affect the value of z. The principal value of $\arg z$ is that which satisfies

$$-\pi < \arg z \leqq \pi.$$

(This is, of course, only a convention. Ambiguity may be equally well, and sometimes more conveniently, avoided by assuming that $0 \leqq \arg z < 2\pi$. This is the convention adopted in Chapter XXII in dealing with conjugate functions. The question which has to be decided in any practical case is whether the values of n other than zero have any useful significance.)

Multiplication of Complex Numbers.

If OP_1 be multiplied by the complex number

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$$

its length is increased in the ratio r_1 and its phase is advanced by θ_1. If the product be then multiplied by

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$$

the length is increased to r_1r_2 and the phase further advanced by the angle θ_2. The modulus of the product is r_1r_2 and the argument is $\theta_1 + \theta_2$. Hence

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + j\sin\theta_1).r_2(\cos\theta_2 + j\sin\theta_2) \\ &= r_1r_2\{\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)\}. \quad . \quad . \quad . \quad (6) \end{aligned}$$

Arithmetical Interpretation of j.

A simple arithmetical interpretation of j may be obtained from equation (6). Treating j as a simple coefficient the equation

$$(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) = \cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)$$

becomes

$$\cos\theta_1\cos\theta_2 + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) + j^2\sin\theta_1\sin\theta_2$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

or

$$j^2\sin\theta_1\sin\theta_2 = -\sin\theta_1\sin\theta_2$$

$$j^2 = -1,\ j = \sqrt{-1} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The Exponential Function.

The exponential function e is defined by the equation

$$e = \underset{n \to \infty}{\text{Limit}} \left(1 + \frac{1}{n}\right)^n \quad . \quad . \quad . \quad . \quad (8)$$

By the binomial theorem one obtains

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1\left(1-\frac{1}{n}\right)}{\underline{|2}} + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \text{etc.}$$

When n is made infinitely great this gives

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots \quad (9)$$

If the expression $\left(1+\frac{1}{n}\right)^{nx}$ be similarly expanded and then n be made infinite the following result is obtained

$$\underset{n \to \infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots$$

Since $\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^n\right\}^x$ it follows that

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots \quad (10)$$

This is the exponential theorem.

(e is clearly incommensurable; its value is 2·71828 . . .)

A simple extension of this theorem is useful. If a be any number, let $c = \log_e a$, then $\varepsilon^c = a$ and

$$a^x = (\varepsilon^c)^x = \varepsilon^{cx} = \varepsilon^{x \log_e a} \text{ whence}$$

$$a^x = 1 + x \log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \ldots \quad (11)$$

The Logarithmic Expansion.

If in equation (11) one writes $1 + y$ for a one obtains

$$(1+y)^x = 1 + x \log_e (1+y) + \frac{x^2}{\underline{|2}}\{\log_e (1+y)\}^2 +, \text{ etc.},$$

which may be written

$$\frac{(1+y)^x - 1}{x} = \log_e (1+y) + x.R$$

where R is a quantity which is not infinite when $x = 0$.

If x is made equal to zero the right-hand member becomes $\log_e (1+y)$.

The left-hand member may be written

$$\frac{(1+y)^x-1}{x}=\frac{1}{x}\left\{1+xy+\frac{x(x-1)}{\underline{|2}}y^2+\frac{x(x-1)(x-2)}{\underline{|3}}y^3 \ldots -1\right\}$$

$$= y + \frac{x-1}{\underline{|2}}y^2 + \frac{(x-1)(x-2)}{\underline{|3}}y^3 +, \text{ etc.}$$

When $x = 0$ this becomes

$$y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \ldots$$

This series is convergent when y is numerically not greater than unity. Therefore when $|y| \not> 1$

$$\log_\epsilon (1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} +, \text{ etc.} \quad . \quad . \quad (12)$$

The Complex Number expressed as an Exponential Function.

From equation (6) we have

$$(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) = \cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)$$

Extending this and taking the particular case of $\theta_1 = \theta_2 = \theta_3$, etc., the result is obtained

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta \quad . \quad . \quad (13)$$

n being a positive integer. The extension to negative integers and to fractions is simple, and it may be accepted that the equation is universally true. Let $\theta = 1$ radian, then equation (13) becomes

$$(\cos 1 + j\sin 1)^n = \cos n + j\sin n.$$

Let k stand for $\cos 1 + j\sin 1$ then $k^n = \cos n + j\sin n$. Putting n first equal to θ and then to $-\theta$

$$k^\theta = \cos\theta + j\sin\theta$$
$$k^{-\theta} = \cos\theta - j\sin\theta$$
$$2j\sin\theta = k^\theta - k^{-\theta}$$

But from equation (11)

$$k^\theta - k^{-\theta} = 2\left\{\theta\log_\epsilon k + \frac{\theta^3(\log_\epsilon k)^3}{\underline{|3}} + \ldots\right\}$$

Hence $j\dfrac{\sin\theta}{\theta} = \log_\epsilon k + \theta^2(R)$ where R is finite. This is true for all values of θ and if θ be made indefinitely small $\dfrac{\sin\theta}{\theta} \to 1$ and the result is obtained

$$\log_e k = j \text{ or } k = e^j$$

whence

$$k^\theta = \cos\theta + j\sin\theta = e^{j\theta} \quad . \quad . \quad . \quad . \quad (14)$$

Therefore any complex number z of modulus r and argument θ can be expressed as

$$z = re^{j\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This form of the complex number is often of great convenience, especially when it has to be differentiated.

From equation (14) the exponential forms of the sine and cosine functions are obtained

$$\cos\theta + j\sin\theta = \varepsilon^{j\theta}$$
$$\cos\theta - j\sin\theta = \varepsilon^{-j\theta}$$

giving

$$\left.\begin{aligned} \cos\theta &= \frac{\varepsilon^{j\theta} + \varepsilon^{-j\theta}}{2} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \ldots \\ \text{and} \quad \sin\theta &= \frac{\varepsilon^{j\theta} - \varepsilon^{-j\theta}}{2j} = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \ldots \end{aligned}\right\} \quad (16)$$

The Hyperbolic Functions.

A circle of radius a (Fig. 4A) may be imagined described by the rotation of a radius vector r of constant length equal to a. The equation to this circle is $x^2 + y^2 = a^2$. As the radius vector rotates from its initial position OA to a position such as OR it sweeps through an angle θ. This angle may be con-

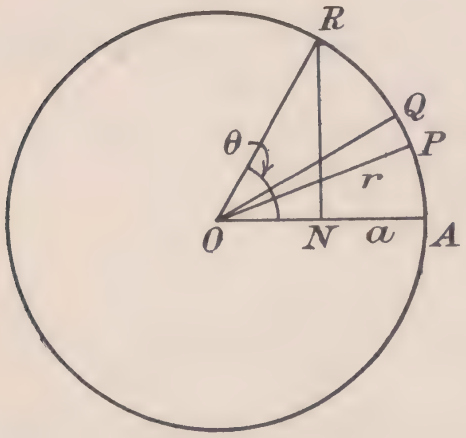


FIG. 4A.

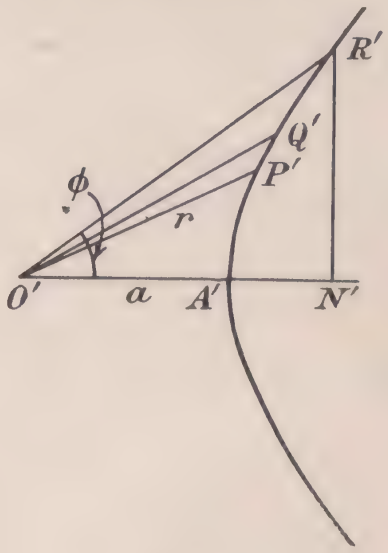


FIG. 4B.

sidered as made up of a large number of infinitesimal parts such as POQ. The small angle POQ may be alternatively defined as (a) the small arc PQ divided by the radius vector r ($= a$ in this case) or (b) twice the area of the sector POQ

divided by a^2. The total angle θ is made up of the sum of all such elements. The angle so determined is in circular radians.

Hyperbolic angles are defined with respect to the equilateral hyperbola $x^2 - y^2 = a^2$, in much the same way as circular angles are defined with respect to the circle $x^2 + y^2 = a^2$. The total hyperbolic angle ϕ (Fig. 4B) may be considered as made up of a large number of elements such as $P'O'Q'$. This small angle $P'O'Q'$ may be alternatively defined as (assuming that $P'Q'$ is very short) (a) the small arc $P'Q'$ divided by the radius vector r, or (b) twice the area of the sector $P'O'Q'$ divided by a^2. (It is shown in Chapter IV that these two definitions are equivalent.) Adding all such elements gives the total angle ϕ. So determined the angle is given in hyperbolic radians.

The circular functions are:

$$\cos\theta = \frac{ON}{OA}\,; \quad \sin\theta = \frac{RN}{OA}$$

The hyperbolic functions are:

$$\cosh\phi = \frac{O'N'}{O'A'}\,; \quad \sinh\phi = \frac{R'N'}{O'A'}$$

It is also shown in Chapter IV that the above definitions give

$$\cosh\phi = \frac{\varepsilon^{\phi} + \varepsilon^{-\phi}}{2} \text{ and } \sinh\phi = \frac{\varepsilon^{\phi} - \varepsilon^{-\phi}}{2}$$

whence

$$\left.\begin{aligned} \cosh\phi &= \frac{e^{\phi} + e^{-\phi}}{2} = 1 + \frac{\phi^2}{\lfloor 2} + \frac{\phi^4}{\lfloor 4} + \frac{\phi^6}{\lfloor 6} + \cdots \\ \sinh\phi &= \frac{e^{\phi} - e^{-\phi}}{2} = \phi + \frac{\phi^3}{\lfloor 3} + \frac{\phi^5}{\lfloor 5} + \cdots \end{aligned}\right\} \quad (17)$$

Examples.

1. Show that $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$\sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

giving $\cosh^2 x - \sinh^2 x = 1$.

2. Express $\cos jx$ and $\sin jx$ as hyperbolic functions.

$$\cos jx = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

$$\sin jx = \frac{e^{-x} - e^{x}}{2j} = j\frac{e^{x} - e^{-x}}{2} = j \sinh x$$

3. Expand $\cosh (A + B)$

$$\cosh(A+B) = \left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right) + \left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$

$$= \cosh A \cosh B + \sinh A \sinh B.$$

4. Expand $\cosh (A + jB)$

$$\cosh (A + jB) = \cosh A \cosh jB + \sinh A \sinh jB$$

$$= \cosh A \cos B + j \sinh A \sin B$$

5. Show that $\log \dfrac{x + \sqrt{x^2 - a^2}}{a} = \cosh^{-1} \dfrac{x}{a}$.

If $$\phi = \log \frac{x + \sqrt{x^2 - a^2}}{a} \text{ then}$$

$$\varepsilon^{\phi} = \frac{x + \sqrt{x^2 - a^2}}{a}$$

Also $$\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)\left(\frac{x - \sqrt{x^2 - a^2}}{a}\right) = 1$$

$$\therefore e^{-\phi} = \frac{x - \sqrt{x^2 - a^2}}{a}$$

Hence $$e^{\phi} + e^{-\phi} = \frac{2x}{a}$$

or $$\frac{x}{a} = \frac{e^{\phi} + e^{-\phi}}{2} = \cosh \phi$$

whence $$\phi = \log \frac{x + \sqrt{x^2 - a^2}}{a} = \cosh^{-1} \frac{x}{a}$$

6. Express $\log (a + jb)$ in the form $A + jB$

$$(a + jb) = r(\cos \alpha + j \sin \alpha) = re^{j\alpha}$$

where $$r^2 = a^2 + b^2 \text{ and } \tan \alpha = \frac{b}{a}$$

$$\log (a + jb) = \log re^{j\alpha} = \log r + j\alpha$$

or $$\log (a + jb) = \log \sqrt{a^2 + b^2} + j \tan^{-1} \frac{b}{a}$$

7. Express $\log(-c)$ in the form $A + jB$.

In Example 6 write $a = -c,\ b = 0$

giving $$\log(-c) = \log c + j\pi$$

8. Express $\cosh^{-1}(-x)$ in the form $\alpha + j\beta$; $x > 1$

$$\cosh^{-1}(-x) = \alpha + j\beta$$

$$\cosh(\alpha + j\beta) = -x = \cosh\alpha\cos\beta + j\sinh\alpha\sin\beta$$

$$\cosh\alpha\cos\beta = -x$$

$$\sinh\alpha\sin\beta = 0.$$

To satisfy the second relation either $\sinh\alpha = 0$ or $\sin\beta = 0$. If $\sinh\alpha$ were zero, $\cosh\alpha$ would be 1 and $\cos\beta$ would be $-x$ which it cannot be if β is real. Therefore $\sin\beta$ must be zero or $\beta = n\pi$. If n were even $\cos\beta$ would be $+1$ and $\cosh\alpha$ would be negative which it cannot be if α is real. Conditions are satisfied if β is an odd multiple of π. The principle value of β is π. Using this value gives $\alpha = \cosh^{-1}x$. Hence

$$\cosh^{-1}(-x) = \cosh^{-1}x + j\pi.$$

9. Express $\cosh^{-1}(x + jy)$ in the form $a + jb$

$$\cosh(a + jb) = x + jy = \cosh a\cos b + j\sinh a\sin b$$

$$\cosh a\cos b = x$$

$$\sinh a\sin b = y$$

$$(1 + x)^2 = 1 + 2\cosh a\cos b + \cosh^2 a\cos^2 b$$

$$y^2 = \sinh^2 a\sin^2 b = (\cosh^2 a - 1)(1 - \cos^2 b)$$

$$= \cosh^2 a + \cos^2 b - \cosh^2 a\cos^2 b - 1$$

$$(1+x)^2+y^2=\cosh^2 a+\cos^2 b+2\cosh a\cos b=(\cosh a+\cos b)^2$$

Similarly

$$(1 - x)^2 + y^2 = (\cosh a - \cos b)^2$$

Hence $$\cosh a = \frac{\sqrt{(1+x)^2+y^2} + \sqrt{(1-x)^2+y^2}}{2}$$

and $$\cos b = \frac{\sqrt{(1+x)^2+y^2} - \sqrt{(1-x)^2+y^2}}{2}$$

whence

$$\cosh^{-1}(x + jy) = \cosh^{-1}\left[\frac{\sqrt{(1+x)^2+y^2} + \sqrt{(1-x)^2+y^2}}{2}\right]$$

$$+ j\cos^{-1}\left[\frac{\sqrt{(1+x)^2+y^2} - \sqrt{(1-x)^2+y^2}}{2}\right]$$

CHAPTER II

PRINCIPLES OF DIFFERENTIATION AND INTEGRATION

Introduction.

The basis of phenomena is change ; the understanding of phenomena consists in recognising the essential rationality, the interdependence of the variables comprising the phenomena. In analysing any particular phenomenon certain variables may be considered as independent and conditioning other, dependent, variables. The distinction is often but a matter of convenience and must not be assumed to imply the order of any causal relation between them. Thus force may be considered as an independent variable and acceleration as the variable dependent upon it, but the positions will normally be reversed in estimating the stresses called into play when a vehicle is braked. From an analytical point of view the decision is made when y is expressed as a function of x, or x as a function of y.

To fix attention a particular case will be taken, that of a body moving with a velocity which may be expressed as

$$u = 100 + 0{\cdot}6t^2 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The body first comes under observation when $t = 5$ and continues to be observed until $t = 10$. It will be imagined that for the purpose of measurement the process may be repeated as many times as desired. There are two observers A and B ; A is provided with a watch and scale, he wishes to determine the velocity at $t = 5$; B travels with the body and he is provided with a watch and speedometer ; he wishes to find the distance travelled between $t = 5$ and $t = 10$.

Differentiation. Observer A.

Observer A wishes to determine the velocity of the body at the instant it first comes under observation, that is at $t = 5$. He selects a definite interval of time Δt, in this case 5 seconds,

and finds that the body travels a distance Δs, in this case 675 cm. His first estimate of the velocity $\Delta s/\Delta t$ is 135 cm. per second. He recognises that this is only the average value over the interval and he can see that the speed is increasing. In order, therefore, to determine the velocity at $t = 5$ he selects a shorter interval. He finds that between $t = 5$ and $t = 6$ the body travels 118·2 cm. He is justified in assuming that 118·2 cm. per second is a closer estimate of the velocity at $t = 5$. He obtains a still better estimate by finding the very short distance δs travelled in a very short time δt. He finds that as δt becomes less and less δs becomes shorter and shorter, but the ratio $\delta s/\delta t$ remains finite and approaches a definite limit. By plotting his results he is able to determine the ratio when δs and δt both become vanishingly small. They are then written ds and dt. This limiting ratio gives him 115 cm. per second as the velocity when $t = 5$. Expressed generally the relations are

$$u = \frac{ds}{dt} = \underset{\delta t \to 0}{\text{Limit}} \left(\frac{\delta s}{\delta t}\right) \quad . \quad . \quad . \quad . \quad (19)$$

ds and dt are called the *differentials* of s and t respectively, $\frac{ds}{dt}$ is the *differential coefficient* (or *first derivative* or *first derived function*) of s with respect to t. It is regarded as the result of operating upon s with the operator $\frac{d}{dt}$. $\frac{ds}{dt}$ is the rate of increase of s with respect to t.

When s can be expressed as a function of t it is possible to determine $\frac{ds}{dt}$. If $s = f(t)$ then upon a small increase δt in t there is a consequent increase δs in s (of course this will be negative if s is decreasing) and so $s + \delta s = f(t + \delta t)$ whence $\delta s = f(t + \delta t) - f(t)$.

Hence

$$\frac{ds}{dt} = \underset{\delta t \to 0}{\text{Limit}} \left(\frac{\delta s}{\delta t}\right) = \underset{\delta t \to 0}{\text{Limit}} \left\{\frac{f(t + \delta t) - f(t)}{\delta t}\right\} \quad . \quad (20)$$

If the observer obtains sufficient measurements connecting the distance travelled and the time he can express the relation between s and t in an equation. This equation will differ according to the point from which distances are measured. If he measures from the point where the body first comes

under observation he finds that he can express the relation as

$$s = 100t + 0{\cdot}2t^3 - 525 \quad . \quad . \quad . \quad . \quad (21)$$

When t increases by δt, s increases by δs and

$$s + \delta s = 100(t + \delta t) + 0{\cdot}2(t + \delta t)^3 - 525$$

giving $$\delta s = 100\delta t + 0{\cdot}2(3t^2.\delta t + 3t.\delta t^2 + \delta t^3)$$

(*Note.*—δt^2 is the square of δt—δt is a single symbol.)

Hence $$\frac{\delta s}{\delta t} = 100 + 0{\cdot}2(3t^2 + 3t.\delta t + \delta t^2)$$

and $$\frac{ds}{dt} = \underset{\delta t \to 0}{\text{Limit}}\left(\frac{\delta s}{\delta t}\right) = 100 + 0{\cdot}6t^2$$

This is the general equation for the velocity at any time and gives the value 115 at $t = 5$. It is an equation more readily determined from the speedometer carried by observer B.

Integration. Observer B.

At $t = 5$, B observes that his speedometer is reading 115 cm. per second and he estimates that in the next 5 seconds he will travel 575 cm. He soon realises that the distance has been underestimated, the indication of the speedometer is increasing. He decides to read the speedometer at the commencement of each second and obtains an estimate of 653 cm. Still unsatisfied, he proceeds with readings taken at the commencement of very short time intervals δt, so short that during one of them the speed has not time to change by an appreciable amount. The distance δs travelled in each interval δt is taken to be $u\delta t$ where u is the velocity at the commencement of the interval. (Note that A and B are carrying out reciprocal operations—B is determining δs by expressing it as $u\delta t$; A is determining u by expressing it as $\delta s/\delta t$.) The total distance travelled approximates to the sum of all the little increments δs between the limits of time t_1 (= 5 seconds) and t_2 (= 10 seconds) and is written as

$$s = \sum_{t_1}^{t_2} \delta s = \sum_{t_1}^{t_2} u\delta t.$$

The exact result is obtained when the interval of time δt is made vanishingly small. The summation is then written $\int_{t_1}^{t_2} u\,dt$, the integral of u between the limits t_1 and t_2.

Differentiation and Integration. Reciprocal relations.

We see that if the velocity

$$u = \frac{ds}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

then the distance s travelled between the time t_1 and t_2 is

$$s = \int_{t_1}^{t_2} u dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The process of integration therefore consists in finding a function which, on being differentiated, gives the function to be integrated.

B, with his speedometer, finds that his readings can be expressed in the form

$$u = 100 + 0{\cdot}6t^2$$

He finds that the function $100t + 0{\cdot}2t^3$ on being differentiated gives u, so does the expression $100t + 0{\cdot}2t^3 - 525$ or in fact $100t + 0{\cdot}2t^3 +$ any constant. This is understandable. The distance s may be measured from any point in the path. That, however, makes no difference to the distance travelled between the times t_1 and t_2. That is determined by substituting t_2 in the expression and subtracting from it the result of substituting t_1 in the expression. Thus

$$\left[s\right]_{t_1}^{t_2} = \left[100t + 0{\cdot}2t^3 + \text{constant}\right]_{t_1}^{t_2}$$
$$= 100(t_2 - t_1) + 0{\cdot}2(t_2^3 - t_1^3)$$

If $t_2 = 10$ and $t_1 = 5$ then $\left[s\right]_5^{10} = 675$ cm.

The briefest examination shows that although attention has been fixed upon the particular case of the relations between time, distance and velocity, the physical significance of those quantities has never been used and the relations given in equations (22) and (23) are quite general and we may write

$$\left.\begin{aligned} \text{if} \quad & y = \int w dx \\ \text{then} \quad & w = \frac{dy}{dx} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

These relations and a further interpretation of the process

of integration are illustrated by Figs. 5A and 5B. Consider a curve $w = \phi(x)$ (Fig. 5A). It is required to determine the area between it and the axis of x between the values of x_a and x_b, that is the area $ABCD$. Consider the vertical strip

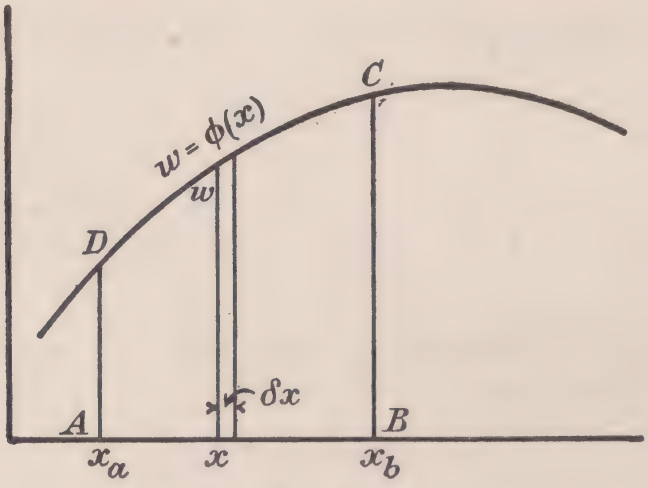


FIG. 5A.

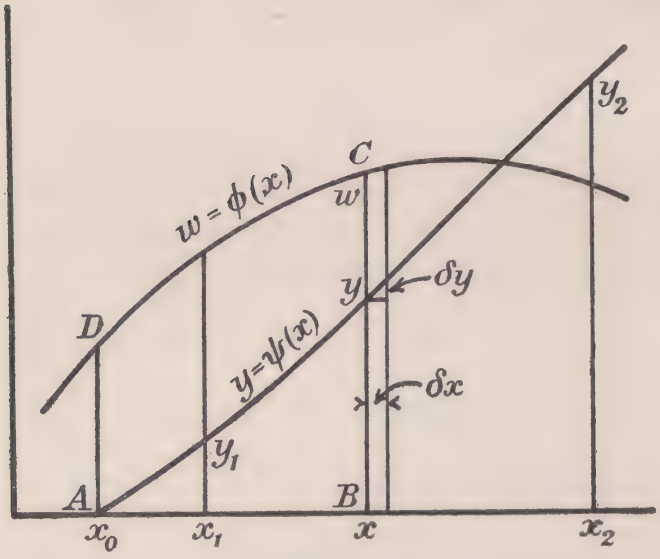


FIG. 5B.

between the abscissæ x and $x + \delta x$. Its height is w; its area $w\delta x$. But the area $ABCD$ is the sum of all such strips between x_a and x_b or the area $= \int_{x_a}^{x_b} w dx$. This curve is redrawn in Fig. 5B together with a second curve $y = \psi(x) = \int_{x_0}^{x} w dx$, x_0 being any arbitrary point on the axis of x. That is to say, at any abscissa x the ordinate y is numerically equal to the area ABCD. If x be increased by δx the area below w is

increased by $w\delta x$ which is therefore the increase in y, namely δy.

$$w\delta x = \delta y \text{ or } w = \frac{\delta y}{\delta x} \text{ the slope of } y.$$

These are, of course, only approximate relations until δx is made vanishingly small, when the relation becomes exact

$$w = \frac{dy}{dx} = \frac{d}{dx}\{\psi(x)\} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Clearly from the diagram

$$\int_{x_1}^{x_2} w dx = y_2 - y_1 = \psi(x_2) - \psi(x_1) \quad . \quad . \quad . \quad (26)$$

The values of y_1 and y_2 change with change of x_0, but their difference is unaltered.

CHAPTER III

METHODS AND RESULTS OF DIFFERENTIATION

Compound Functions.

Before proceeding to the actual process of differentiation certain rules for compound functions may be given.

(*a*) *Added Constant.*—If $y = u + c$ where c is a constant then $\frac{dy}{dx} = \frac{du}{dx}$ since any change in y is wholly confined to the change in u. The differential coefficient of a constant is zero.

(*b*) *Constant Multiplier.*—If $y = cu$ where $u = f(x)$ and c a constant then

$$\frac{dy}{dx} = *\text{Lt}\,\frac{c\,.f(x+\delta x) - c\,.f(x)}{\delta x} = c\,.\,\text{Lt}\,\frac{f(x+\delta x) - f(x)}{\delta x} = c\frac{du}{dx}$$

or

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

(*c*) *Sum or Difference.*

If $y = u + v$ where $u = \phi(x)$, $v = \psi(x)$

$$\text{then } \frac{dy}{dx} = \text{Lt}\,\frac{\phi(x+\delta x) + \psi(x+\delta x) - \phi(x) - \psi(x)}{\delta x}$$

$$= \text{Lt}\,\frac{\phi(x+\delta x) - \phi(x)}{\delta x} + \text{Lt}\,\frac{\psi(x+\delta x) - \psi(x)}{\delta x}$$

$$= \frac{du}{dx} + \frac{dv}{dx}$$

Extending it is seen that

$$\frac{d}{dx}(u \pm v \pm w \pm \ldots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \ldots$$

(*d*) *Product of Functions.*

If $y = uv$ where $u = \phi(x)$, $v = \psi(x)$

$$\text{then } \frac{dy}{dx} = \text{Lt}\,\frac{\phi(x+\delta x)\psi(x+\delta x) - \phi(x)\psi(x)}{\delta x}$$

* Lt when not otherwise specified stands for $\underset{\delta x \longrightarrow 0}{\text{Limit}}$

Adding and subtracting $\phi(x+\delta x)\psi(x)/\delta x$ this becomes

$$\frac{dy}{dx} = \text{Lt } \phi(x+\delta x)\frac{\psi(x+\delta x)-\psi(x)}{\delta x} + \text{Lt } \psi(x)\frac{\phi(x+\delta x)-\phi(x)}{\delta x}$$

$$= \phi(x)\frac{d}{dx}[\psi(x)] + \psi(x)\frac{d}{dx}[\phi(x)]$$

or $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

If $$y = uvw$$

then $$\frac{dy}{dx} = u\frac{d}{dx}(vw) + vw\frac{du}{dx}$$

$$= u\left(v\frac{dw}{dx} + w\frac{dv}{dx}\right) + vw\frac{du}{dx}$$

or $$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}.$$

(*e*) *Quotient.*

If $$y = \frac{u}{v} \text{ then } u = vy$$

and $$\frac{du}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}$$

or $$\frac{dy}{dx} = \frac{1}{v}\left(\frac{du}{dx} - y\frac{dv}{dx}\right) = \frac{1}{v}\left(\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

(*f*) *Function of a Function.*

If $y = \phi(u)$ where $u = \psi(x)$ then $y = \phi[\psi(x)]$.

If x be increased by δx then u increases by δu and y increases by δy.

Clearly $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x} \text{ and in the limit}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

(*g*) *Inverse Function.*

If $y = \sin^{-1} x$, or any similar inverse function it may be more convenient to express the equation in the direct form,

such as $x = \sin y$. The differential coefficient $\frac{dx}{dy}$ may then be found. As in (*f*),

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}.$$

(*h*) *Logarithmic Differentiation.*

If y is expressed as $y = u^v$ where u and v are functions of x it may be convenient to rewrite the equation as $\log y = v \log u$ and then use (*f*).

Differentiation of Standard Forms.

The ordinary mathematical functions are composed of a number of fundamental forms, the algebraical function x^n, the exponential function ε^x, the logarithmic function $\log_\epsilon x$, the direct circular functions $\sin x$ etc., the direct hyperbolic functions $\sinh x$, etc., the inverse circular functions $\sin^{-1} x$ (or arc $\sin x$), etc., and the inverse hyperbolic functions $\sinh^{-1} x$ (or arc $\sinh x$), etc. It is proposed to derive the differential coefficients of three of these from the definition, namely x^n, ε^x and $\sin x$. The others given in the table in Appendix I can be derived by the reader.

Differentiation of x^n.

$$\text{If } y = x^n, \quad \frac{dy}{dx} = \text{Lt}\, \frac{(x + \delta x)^n - x^n}{\delta x} = \text{Lt}\, x^n \frac{\left(1 + \dfrac{\delta x}{x}\right)^n - 1}{\delta x}$$

$$= \text{Lt}\, x^n \frac{\left\{1 + n\dfrac{\delta x}{x} + \left(\dfrac{\delta x}{x}\right)^2 R\right\} - 1}{\delta x}$$

As the series within the brackets is convergent when δx is small, the remainder R is finite, and hence

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Differentiation of e^x.

$$\text{If } y = e^x, \quad \frac{dy}{dx} = \text{Lt}\, \frac{e^{x+\delta x} - e^x}{\delta x} = \text{Lt}\, \frac{e^x\{e^{\delta x} - 1\}}{\delta x}$$

$$= \text{Lt}\, e^x \frac{\{1 + \delta x + \delta x^2(R)\} - 1}{\delta x}$$

Again the series within the brackets is convergent and R is finite. Hence $\frac{d}{dx}(e^x) = e^x$.

Differentiation of sin x.

$$\text{If } y = \sin x, \quad \frac{dy}{dx} = \text{Lt}\, \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

$$= \text{Lt}\, \frac{2 \cos\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x}$$

When δx is made vanishingly small $\sin \frac{\delta x}{2} = \frac{\delta x}{2}$ and

$$\frac{d}{dx}(\sin x) = \cos x.$$

Examples illustrating the Rules.

The following examples are worked to illustrate the application of the rules given.

(*a*) *Product.*

$$y = x^n \sin x, \quad \frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(x^n \sin x) = nx^{n-1} \sin x + x^n \cos x$$

(*b*) *Quotient.*

$$y = \frac{x^n}{\sin x}, \quad \frac{dy}{dx} = \frac{nx^{n-1} \sin x - x^n \cos x}{\sin^2 x}$$

or

$$\frac{d}{dx}\left(\frac{x^n}{\sin x}\right) = \frac{x^{n-1}}{\sin x}\{n - x \cot x\}$$

(*c*) *Function of a Function.*

$$y = \sin x^n, \quad u = x^n, \quad y = \sin u$$

$$\frac{du}{dx} = nx^{n-1}, \quad \frac{dy}{du} = \cos u = \cos x^n$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nx^{n-1} \cos x^n$$

After a little practice, expressions of this type can usually be differentiated without actually making the substitutions. Thus in the particular case, first differentiate the sine obtaining

$\cos x^n$, then differentiate the x^n obtaining nx^{n-1} and the product $nx^{n-1}\cos x^n$ gives $\dfrac{dy}{dx}$.

(*d*) *Inverse Function.*

(i) $$y = \sin^{-1} x, \quad x = \sin y$$

$$\frac{dx}{dy} = \cos y = \cos(\sin^{-1} x) = \sqrt{1 - x^2}$$

giving $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

(ii) $$y = \log_\varepsilon x, \quad x = \varepsilon^y$$

$$\frac{dx}{dy} = \varepsilon^y = x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

(*e*) *Logarithmic Differentiation.*

$$y = (\log x)^{\sin x}$$

$$\log y = \sin x \,.\, \log(\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \,.\, \log(\log x) + \sin x \,\frac{1}{\log x} \,.\, \frac{1}{x}$$

or $$\frac{dy}{dx} = (\log x)^{\sin x}\left\{\cos x \,.\, \log(\log x) + \frac{\sin x}{x \log x}\right\}$$

(*f*) *Further Examples involving a Combination of the Rules.*

(i) $$y = e^{-\frac{R}{2L}t}\cos(\omega t + \phi)$$

$$\frac{dy}{dt} = -\frac{R}{2L}e^{-\frac{R}{2L}t}\cos(\omega t + \phi) - \omega e^{-\frac{R}{2L}t}\sin(\omega t + \phi)$$

(ii) $$y = \log\frac{1 - x}{1 + x}$$

$$\frac{dy}{dx} = \frac{1 + x}{1 - x} \,.\, \frac{-(1 + x) - (1 - x)}{(1 + x)^2} = -\frac{2}{1 - x^2}$$

(iii) $$y = \sin^2 e^x$$

$$\frac{dy}{dx} = 2 \sin e^x \,.\, \cos e^x \,.\, e^x = e^x \sin 2e^x$$

Successive Differentiation.

When a function has been differentiated, giving the *first derivative*, this first derivative may be differentiated giving the

second derivative. The process may be continued indefinitely. The usual notation for $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is $\frac{d^2y}{dx^2}$ and for $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is $\frac{d^3y}{dx^3}$, etc. Other convenient expressions for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots \frac{d^ny}{dx^n}$ are (if $y = f(x)$),

$$\begin{array}{l} f'(x),\ f''(x),\ f'''(x) \ \ldots\ f^n(x) \\ y_1\ ,\ y_2\ ,\ y_3 \quad \ldots\ y_n \\ Dy\ ,\ D^2y\ ,\ D^3y \quad \ldots\ D^ny \end{array}$$

It is common for a function to become more and more complicated as it is successively differentiated, though this is not a universal rule.

In some cases a rearrangement of the result of differentiation will indicate a general method which will permit the nth derivative to be written down from inspection. Thus if

$$y = A \sin(\omega t + \phi)$$

$$\frac{dy}{dt} = \omega A \cos(\omega t + \phi) = \omega A \sin\left(\omega t + \phi + \frac{\pi}{2}\right)$$

The effect of differentiation is thus to increase the angle by $\frac{\pi}{2}$ and to multiply the coefficient by ω. It therefore follows that

$$\frac{d^ny}{dt^n} = \omega^n A \sin\left(\omega t + \phi + \frac{n\pi}{2}\right)$$

As a particular case $\frac{d^4y}{dt^4} = \omega^4 A \sin(\omega t + \phi + 2\pi) = \omega^4 y$

Hence $y = A \sin(\omega t + \phi)$ is a solution of the *differential equation* $\frac{d^4y}{dt^4} = \omega^4 y$.

Similar changes obviously occur when $A \cos(\omega t + \phi)$ is differentiated.

Another case of interest is $y = Ae^{bt} \sin(\omega t + \phi)$

$$\frac{dy}{dt} = bAe^{bt} \sin(\omega t + \phi) + \omega Ae^{bt} \cos(\omega t + \phi)$$

$$= A\sqrt{\omega^2 + b^2}e^{bt}\left\{\frac{b}{\sqrt{\omega^2 + b^2}} \sin(\omega t + \phi) + \frac{\omega}{\sqrt{\omega^2 + b^2}} \cos(\omega t + \phi)\right\}$$

If $r = \sqrt{\omega^2 + b^2}$ and $\theta = \tan^{-1} \omega/b$ then

$$\frac{dy}{dt} = Are^{bt}\{\cos\theta \sin(\omega t + \phi) + \sin\theta \cos(\omega t + \phi)\}$$

$$= Are^{bt} \sin(\omega t + \phi + \theta)$$

Hence

$$\frac{d^n y}{dt^n} = Ar^n e^{bt} \sin(\omega t + \phi + n\theta)$$

Successive Differentiation of a Product. Leibnitz's Theorem.

If

$$y = uv$$
$$y_1 = u_1 v + uv_1 \text{ (p. 19)}$$
$$y_2 = u_2 v + 2u_1 v_1 + uv_2$$
$$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + uv_3$$

Similarly

$$y_n = u_n v + nu_{n-1}v_1 + \frac{n(n-1)}{\lfloor 2}u_{n-2}v_2 \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}u_{n-r}v_r \ldots uv_n$$

This is Leibnitz's theorem. It can be proved by induction as follows. Clearly it is true when $n = 1$, 2 and 3. If it can be shown that by differentiating y_n the same expansion is obtained except that $n + 1$ is substituted for n the theorem is proved.

Assuming the theorem to be true, the coefficient of $u_{n-r+1}v_r$ in the expansion of y_{n+1} should be

$$\frac{(n+1).n.(n-1)\ldots(n-r+2)}{\lfloor r}$$

The term in $u_{n-r+1}v_r$ in the expansion of y_{n+1} arises from the differentiation of the following two terms in the expansion of y_n:

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{\lfloor r-1}u_{n-r+1}v_{r-1}$$

and

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}u_{n-r}v_r$$

Terms in $u_{n-r+1}v_r$ are given when the former is differentiated with respect to v and the latter with respect to u. The

coefficient of $u_{n-r+1}v_r$ in the expansion of y_{n+1} is therefore

$$\frac{n(n-1)\ldots(n-r+2)}{\lfloor r-1} + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}$$

$$= \frac{n(n-1)\ldots(n-r+2)}{\lfloor r}\{r+n-r+1\}$$

$$= \frac{(n+1).n(n-1)\ldots(n-r+2)}{\lfloor r}$$

The theorem is therefore proved.

Maclaurin's Theorem.

It is often possible to express a function in a convergent series of ascending powers of the variable. Such a series is of considerable use in computing values of the function. For instance, when $x \not> 1$, $(1-x)^{-1} = 1 + x + x^2 + x^3$, etc. . . . Such a series would not in the general case be used for computing $(1-x)^{-1}$, though in particular cases it may be used. Or again $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7}$, etc. (equation 16, p. 8). This series is evidently a useful one for computation.

If it be assumed that a particular function of x may be so expanded then

$$f(x) = a + bx + cx^2 + dx^3 + ex^4, \text{ etc.}$$

and

$$\begin{aligned} f'(x) &= b + 2cx + 3dx^2 + 4ex^3, \text{ etc.} \\ f''(x) &= 2c + 3.2dx + 4.3ex^2, \text{ etc.} \\ f'''(x) &= 3.2.1d + 4.3.2.ex, \text{ etc.} \\ f''''(x) &= 4.3.2.1.e, \text{ etc.} \end{aligned}$$

These equations are true for all values of x. If x be made zero then the following relations are obtained:

$a = f(0)$, $b = f'(0)$, $c = f''(0)/\lfloor 2$, $d = f'''(0)/\lfloor 3$, $e = f''''(0)/\lfloor 4$, etc., and therefore

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\lfloor 2} + f'''(0)\frac{x^3}{\lfloor 3} + f''''(0)\frac{x^4}{\lfloor 4} + \text{etc.} \quad . \quad (27)$$

This is Maclaurin's series.

Thus if

$f(x)$	$= \sin x$	$f(0)$	$= 0$
$f'(x)$	$= \cos x$	$f'(0)$	$= 1$
$f''(x)$	$= -\sin x$	$f''(0)$	$= 0$
$f'''(x)$	$= -\cos x$	$f'''(0)$	$= -1$
$f''''(x)$	$= \sin x$	$f''''(0)$	$= 0$

Since $f(x) = f''''(x)$ the values are then repeated. The series is thus given as

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \text{etc.} \quad . \quad . \quad (28)$$

This is identical with equation 16, p. 8, but obtained by an entirely different method.

Taylor's Theorem.

By definition $f'(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x}$. When δx is not vanishingly small but has a small finite value h then

$$f'(x) \doteqdot \frac{f(x + h) - f(x)}{h} \quad . \quad . \quad . \quad (29)$$

or

$$f(x + h) \doteqdot f(x) + hf'(x) \quad . \quad . \quad . \quad . \quad (30)$$

In many cases this is sufficiently accurate to determine the small increase in $f(x)$ brought about by an increase in x. In Fig. 6 when $x = OB$ then $f(x) = BA$; when x is increased by $h(= BC)$ to OC then $f(x + h) = CE$. The tangent at A gives the rate of increase of $f(x)$ with respect to x and hence $FD = hf'(x)$. When h is small D nearly coincides with E and equation 30 is sufficiently accurate. It is often convenient to have an expression giving a closer approximation when h has a value which is only moderately small. This is provided by assuming that $f(x + h)$ can be expressed as a series in ascending powers of h, thus

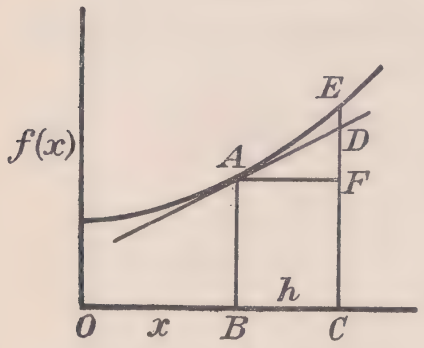


FIG. 6.

$$f(x + h) = P + Qh + Rh^2 + Sh^3 + Th^4 + Uh^5 + \ldots$$

where P, Q, R, etc., are functions of x but not of h. If now the expression be differentiated with respect to h then

$$f'(x + h) = Q + 2hR + 3h^2S + 4h^3T + 5h^4U, \text{etc.}$$

$$f''(x + h) = 2R + \lfloor 3hS + 4 \,.\, 3h^2T + 5 \,.\, 4h^3U, \text{etc.}$$

$$f'''(x + h) = \lfloor 3S + \lfloor 4hT + 5 \,.\, 4 \,.\, 3h^2U, \text{etc.}$$

$$f''''(x + h) = \lfloor 4T + \lfloor 5hU, \text{ etc.}$$

These are still true when $h = 0$ and hence

$P = f(x)$, $Q = f'(x)$, $R = f''(x)/\lfloor 2$, $S = f'''(x)/\lfloor 3$, $T = f''''(x)/\lfloor 4$, etc.,

and hence

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \text{etc.}$$

This is Taylor's series.

Example.—Given $\sin 30° = 0{\cdot}5$ and $\cos 30° = 0{\cdot}8661$, find $\sin 31°$.

$$f(x) = \sin x$$

$$f(x + h) = \sin x + h \cos x - \frac{h^2}{\lfloor 2} \sin x - \frac{h^3}{\lfloor 3} \cos x + \frac{h^4}{\lfloor 4} \sin x \text{ etc.}$$

$$= \sin x\left(1 - \frac{h^2}{\lfloor 2} + \frac{h^3}{\lfloor 4} \ldots\right) + \cos x\left(h - \frac{h^3}{\lfloor 3} \ldots\right)$$

In this case $h = \pi/180$ giving

$$\sin 31° = 0{\cdot}500\left(1 - \frac{0{\cdot}0003}{2} \ldots\right) + 0{\cdot}8661(0{\cdot}01745 \ldots)$$

$$= 0{\cdot}5000 - 0{\cdot}0001 + 0{\cdot}0151 = 0{\cdot}5150$$

A practical application of interest which does not actually invoke Taylor's theorem but only equation 30 is the determination of the approximate solution of an equation. It may be seen that a particular value of x is a rough solution; it is required to determine the solution more exactly. To x must be added a small positive or negative quantity h; if h is correctly chosen then $f(x + h) = 0$. But from equation 30

$$f(x + h) \doteqdot f(x) + hf'(x)$$

and hence

$$h \doteqdot -\frac{f(x)}{f'(x)} \quad \ldots\ldots \quad (31)$$

Example.—It is seen that 2 is a rough solution of the equation $x^3 - 3x - 2{\cdot}9 = 0$. A more exact solution is required.

$$f(x) = x^3 - 3x - 2{\cdot}9 \qquad f(2) = -0{\cdot}9$$

$$f'(x) = 3x^2 - 3 \qquad f'(2) = 9$$

$$h = -\frac{-0{\cdot}9}{9} = 0{\cdot}1 \quad \text{giving} \quad 2{\cdot}1 \quad \text{as a closer}$$

approximation. A still closer approximation is obtained by substituting $2{\cdot}1$ in $f(x)$ and $f'(x)$; thus $f(2{\cdot}1) = 0{\cdot}061$, $f'(2{\cdot}1) = 10{\cdot}23$, $h = -0{\cdot}0060$ giving, to 4 places, $x = 2{\cdot}0940$.

Differentials.

Equation (29) may be written

$$f(x + \delta x) - f(x) \doteqdot \delta x \,.\, f'(x) \quad . \quad . \quad . \quad (32)$$

This equation becomes exactly true as δx approaches the limiting value dx (the differential of x) and so the differential of $f(x)$, namely

$$d\{f(x)\} = f'(x)dx \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Much use of this relation is made in integration. Thus, if $x = \sin\theta$, $dx = \cos\theta d\theta$, and again if $x = \log\theta$, $dx = \frac{1}{\theta}d\theta$.

Maxima and Minima.

In the curve of Fig. 7, y has maximum values at A and C and minimum values at B and D. At each of these points $\frac{dy}{dx} = 0$ and the points can be determined from this equation. Before a maximum point $\frac{dy}{dx}$ is positive, and is negative after,

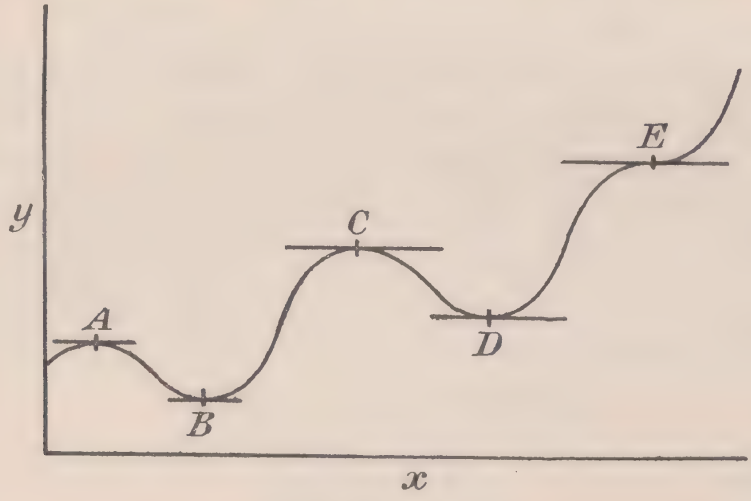


FIG. 7.

and hence at a maximum point $\frac{d^2y}{dx^2}$ is negative. Conversely at a minimum value $\frac{d^2y}{dx^2}$ is positive. This test for a maximum or minimum breaks down if $\frac{d^2y}{dx^2} = 0$. In such cases further tests may be given, but the simplest in practice is to determine the nature of the change in the neighbourhood of the point by giving x first a value a little less and then a little

greater. This test is particularly suitable in dealing with such a point as E at which $\frac{dy}{dx} = 0$ and y has neither a maximum nor a minimum value but a point of inflection. In most practical cases the existence of a maximum or minimum value is known from physical considerations. All that is required is to determine its position and value. The following example illustrates this point.

Example.—The potential of a length of 2 miles of a return rail may be represented by the expression $V = 12{\cdot}6x - 7x^2$ where x is the distance from the home end. Find the maximum potential occurring in this length of rail and its position.

At the home end $V = 0$ and at 2 miles $V = -2{\cdot}8$ volts.

Generally, $\frac{dV}{dx} = 12{\cdot}6 - 14x$. If $\frac{dV}{dx} = 0$, $x = 0{\cdot}9$. At this point 0·9 mile from the home end the potential is a maximum and its value is $12{\cdot}6 \times 0{\cdot}9 - 7 \times 0{\cdot}81 = 5{\cdot}67$ volts.

That it is a *maximum* is easily checked. At $x = 0{\cdot}8$, $V = 5{\cdot}60$ volts and at $x = 1{\cdot}0$, $V = 5{\cdot}60$ volts. The value of V being less on both sides of 0·9 indicates that the value at 0·9 is a *maximum*.

Or again $\frac{d^2V}{dx^2} = -14$. $\frac{d^2V}{dx^2}$ being negative, the point found must be a maximum point.

Example.—If the torque exerted by an induction motor may be expressed as $T = \frac{AsR}{R^2 + s^2X^2}$ where R = rotor resistance, X = standstill rotor reactance, s = slip, A = constant, show that the maximum torque exerted is $T_m = \frac{A}{2X}$ and occurs when the slip is equal to R/X.

$$\frac{dT}{ds} = \frac{AR(R^2 + s^2X^2) - 2sX^2(AsR)}{(R^2 + s^2X^2)^2} = 0$$

giving $R^2 - s^2X^2 = 0$ or $s = R/X$

$$T_m = \frac{A\frac{R^2}{X}}{R^2 + R^2} = \frac{A}{2X}$$

CHAPTER IV

METHODS AND RESULTS OF INTEGRATION

Introduction.

In Chapter II it was shown that the process of integration is identically expressed by the relations

$$\int_{x_1}^{x_2} w dx = \psi(x_2) - \psi(x_1)$$

where $w = \frac{d}{dx}[\psi(x)]$ (equations 25 and 26, p. 17)

In what follows the typical *definite* integral will be written as

$$\left.\begin{aligned} \int_a^b \phi(x)dx &= \psi(b) - \psi(a) \\ \psi'(x) &= \phi(x) \end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

where

When the upper limit is undetermined, as in

$$\int_a^x \phi(x)dx = \psi(x) - \psi(a)$$

the integral is said to be *corrected.*

When neither limit is determined, as in

$$\int \phi(x)dx = \psi(x)$$

the integral is said to be *indefinite* or *uncorrected.*

Constant of Integration.

It is obvious that if $\phi(x) = \frac{d}{dx}[\psi(x)]$ then it is also equal to $\frac{d}{dx}[\psi(x) + c]$ where c is any arbitrary constant. Although not always written down it is understood to exist in all cases of indefinite integration. When the integration has been

made definite the constant disappears since if

$$\int \phi(x)dx = \psi(x) + c \text{ then}$$

$$\int_a^b \phi(x)dx = \psi(b) + c - \psi(a) - c$$

$$= \psi(b) - \psi(a)$$

The physical significance of this point was emphasised in the example treated in Chapter II.

The constant of integration has, however, a real position in analysis. For instance, it may be known from practical considerations that, at a value of $x = a$, $\int \phi(x)dx = X$. Then $\psi(a) + c = X$, giving $c = X - \psi(a)$ and

$$\int \phi(x)dx = \psi(x) + X - \psi(a) \quad . \quad . \quad . \quad (35)$$

It is possible in some circumstances that $\psi(x) + c$ may differ quite definitely in form from $\psi(x)$. Different methods of indefinite integration may give results sufficiently different that at first sight their accuracy may be questioned. An examination shows, however, that they differ only by an added constant. For instance $\int \frac{1}{1+x^2}dx$ may be either $\tan^{-1} x$ or $-\cot^{-1} x$. These two are, of course, not equal but they differ only by a constant since $\tan^{-1} x - (-\cot^{-1} x) = \frac{\pi}{2}$.

General Laws of Integration.

From Chapter III it necessarily follows that

(i) if $\frac{d}{dx}[\psi(x)] = \phi(x)$

then $\int \phi(x)dx = \psi(x) +$ an arbitrary constant . . (36)

(ii) $\int cf(x)dx = c\int f(x)dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$

where c is a constant.

(iii) $\int \{f(x) + \phi(x) + \psi(x)\}dx$

$$= \int f(x)dx + \int \phi(x)dx + \int \psi(x)dx \quad . \quad . \quad . \quad (38)$$

General Processes of Integration.

In Chapter III the principles and methods of differentiation were treated. Though the practice may sometimes be laborious it presents no inherent difficulty and a differentiation can always be performed. With integration the position is different. It is required to find a function which on being differentiated gives the function to be integrated. Often the process employed is little more than intelligent guesswork. It is aimed at reducing the expression to a form which is the differential coefficient of a known function. Sometimes this is impossible; a solution may then be obtained in the form of an infinite series. This case will be deferred until later. The two most potent simple processes are (*a*) the method of substitution and (*b*) integration by parts. These processes will be illustrated after the next section.

Integration of x^n.

If x^n be differentiated the result is nx^{n-1} and again $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$. Hence $\int x^n dx = \frac{x^{n+1}}{n+1}$.

When $n = -1$ this form of the integral becomes infinite and is useless from a practical point of view. A useful form of the integral can be obtained as follows. Assume that $n = -1 + h$, where h is small and is ultimately to be made zero, then

$$\int x^{-1+h} dx = \frac{x^h}{h} + \text{an arbitrary constant}$$

From equation 11, p. 6, this may be written

$$\frac{1 + h \log x + \frac{h^2}{2}(\log x)^2 \text{ etc.}}{h} + \text{an arbitrary constant}$$

$$= \frac{h \log x + \frac{h^2}{2}(\log x)^2 \text{ etc.}}{h} + \text{another constant}$$

$$= \log x + \frac{h}{2}(\log x)^2 \text{ etc.} + \text{constant}$$

Hence when $h = 0$, $\int \frac{1}{x} dx = \log_e x + c$.

This result is, of course, known from the result of differentiating $\log_e x$.

Method of Substitution.

Many expressions which themselves are not standard forms may be rendered so by change of the independent variable. In many cases the substitution to be made almost suggests itself. Other cases are more difficult and the best substitution may not be made at the first attempt. The result of such an attempt may, however, suggest a more effective substitution. In this sense the process is (like division) simply guessing, aided by experience and practice. The method is perhaps best illustrated by a few simple examples. After these some general indications of likely substitutions will be given.

Examples.

1. To determine $\int \frac{1}{x} \cos(\log x) dx$.

Put $u = \log x$, then $du = \frac{1}{x} dx$. Making these substitutions the integral becomes

$$\int \cos u \, du = \sin u = \sin(\log x)$$

2. To determine $\int \frac{dx}{x^2 + 6x + 10}$.

Put $x + 3 = u$, then $du = dx$ and $x^2 + 6x + 10 = u^2 + 1$ and the integral becomes

$$\int \frac{du}{u^2 + 1} = \tan^{-1} u = \tan^{-1}(x + 3).$$

3. To determine $\int \frac{x + 3}{x^2 + 6x + 10} dx$.

As before, put $x + 3 = u$ and the integral becomes $\int \frac{u}{u^2 + 1} du$.

Put $u^2 + 1 = z$, then $2u \, du = dz$ and the integral becomes

$$\frac{1}{2} \int \frac{dz}{z} = \frac{1}{2} \log(u^2 + 1) = \log \sqrt{x^2 + 6x + 10}$$

4. To determine $\int \frac{f'(x)}{f(x)} dx$.

Put $f(x) = u$, then $f'(x)dx = du$ and the integral becomes $\int \frac{du}{u} = \log u = \log f(x)$.

This is an important form. It will be seen that the previous example could have been solved in this way.

Definite Integrals by the Method of Substitution.

When the value of a definite integral is required and the variable is changed, it is not necessary to reconvert the variable after the determination of the integral. The limits may be changed at the same time as the variable. Thus in the definite integral

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}} dx$$

a convenient substitution is $x = \tan\theta$. Then $dx = \sec^2\theta \, . \, d\theta$ and also when $x = 0$, $\theta = 0$ and when $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$. Hence

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}} dx = \int_0^{\frac{\pi}{3}} \frac{\tan\theta \sec^2\theta}{\sqrt{1+\tan^2\theta}} d\theta = \int_0^{\frac{\pi}{3}} \sec\theta \tan\theta \, d\theta$$

$$= \Big[\sec\theta\Big]_0^{\frac{\pi}{3}} = 2 - 1 = 1$$

Convenient Substitutions.

(i) When the quantity to be integrated is a rational function of $\sin\theta$ or $\cos\theta$ it is often helpful to express it as a function of $\frac{\theta}{2}$ and then make a substitution for $\tan\frac{\theta}{2}$. Thus $\int \frac{d\theta}{\sin\theta}$ may be written as

$$\int \frac{d\theta}{\sin\theta} = \int \frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \int \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}} d\theta$$

This is of the form $\frac{f'(\theta)}{f(\theta)}$ where $f(\theta) = \tan\frac{\theta}{2}$.

Hence $\int \frac{d\theta}{\sin\theta} = \log\tan\frac{\theta}{2}$.

(ii) When expressions of the form $a^2 - x^2$ occur the substitution of $a\sin\theta$ for x may be useful.

Thus $\int \sqrt{7 - 6x - x^2} dx = \int \sqrt{16 - (x+3)^2} dx$

Put $x + 3 = 4 \sin \theta$, giving $dx = 4 \cos \theta \, . \, d\theta$ and the integral becomes

$$\int \sqrt{4^2 - 4^2 \sin^2 \theta} \, . \, 4 \cos \theta \, . \, d\theta = 16 \int \cos^2 \theta d\theta$$

$$= 8 \int (1 + \cos 2\theta) d\theta = 8\theta + 4 \sin 2\theta$$

This may be expressed in terms of x if desired.

(iii) When expressions of the form of $x^2 + a^2$ occur it may be helpful to write either (*a*), $x = a \tan \theta$ or (*b*), $x = a \sinh \theta$.

An example of the former substitution has already been given. The same example may be worked out by making the second substitution. Thus if $x = \sinh \theta$, $dx = \cosh \theta \; d\theta$ and

$$\int \frac{x}{\sqrt{1 + x^2}} dx = \int \frac{\sinh \theta \, . \, \cosh \theta \, . \, d\theta}{\sqrt{1 + \sinh^2 \theta}} = \int \sinh \theta \; d\theta$$

$$= \cosh \theta = \sqrt{1 + \sinh^2 \theta} = \sqrt{1 + x^2}$$

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{1 + x^2}} dx = \sqrt{4} - \sqrt{1} = 1$$

(iv) For an expression of the form $x^2 - a^2$ a substitution which is often useful is $x = a \cosh \theta$. Thus in $\int \frac{dx}{\sqrt{x^2 - a^2}}$ put $x = a \cosh \theta$ giving $dx = a \sinh \theta \; d\theta$. The integral then becomes $\int \frac{a \sinh \theta \, d\theta}{a \sinh \theta} = \theta = \cosh^{-1} \frac{x}{a}$.

If $\log \frac{x + \sqrt{x^2 - a^2}}{a}$ be differentiated it will be found that the derivative is $\frac{1}{\sqrt{x^2 - a^2}}$. It therefore follows that $\log \frac{x + \sqrt{x^2 - a^2}}{a}$ is an alternative form, or alternative solution, of the integral. Actually the two expressions are identical, as was shown in example 5 at the end of Chapter I.

Integration by Parts.

It has already been seen that

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

It therefore follows that

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

or

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \quad . \quad . \quad . \quad . \quad (39)$$

That is, if a function to be integrated consists of the product of two terms, one of which is directly integrable, the integral of the function may be connected with a second integral of a product of two terms, these terms being respectively the integral of the integrable original term and the differential coefficient of the other original term. This second integral may be easier or more difficult than the first. The method may be illustrated by a simple example.

Example.—To determine $\int x \log x \, dx$.

Put $u = \log x$ and $\frac{dv}{dx} = x$

then

$$\int x \log x \, dx = \frac{x^2}{2} \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4}$$

. The method may sometimes be useful when the function is one which may readily be differentiated but whose integral is not obvious. Unity is then taken as the term to be integrated.

Example.—To determine $\int \log x \, dx$.

Put $u = \log x$, $\frac{dv}{dx} = 1$.

Then $\int 1 \, . \log x \, . \, dx = x \log x - \int x \, . \frac{1}{x} dx$

$$= x \log x - x$$

When both terms are integrable some care is necessary in the choice of the parts, and until practice is acquired useless substitutions may be made.

Example.—To determine $\int xe^x \, dx$.

If the parts chosen be $u = e^x$ and $\frac{dv}{dx} = x$ then the result obtained is

$$\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx$$

This second integral is more complicated than the original and evidently the solution is not obtainable this way. The correct parts are

$$u = x \text{ and } \frac{dv}{dx} = e^x \text{ then}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x - 1)$$

Integration by parts may be performed several times. The first integration may give an integral which is simpler than the original but still is not directly integrable. Further integration, or integrations, may give the result required.

Example.—To determine $\int x^2 \sin x\,dx$.

Put $u = x^2$, $\frac{dv}{dx} = \sin x$.

Then $$\int x^2 \sin x\,dx = -x^2 \cos x + \int 2x \cos x\,dx.$$

To evaluate $\int 2x \cos x\,dx$, put $u = 2x$, $\frac{dv}{dx} = \cos x$, giving

$$\int 2x \cos x\,dx = 2x \sin x - \int 2 \sin x\,dx$$
$$= 2x \sin x + 2 \cos x$$

whence $$\int x^2 \sin x\,dx = -x^2 \cos x + 2x \sin x + 2 \cos x$$

In some cases an integral of the same form as the original may be obtained after successive integration. This may enable the integral to be obtained.

Example.—To determine $\int e^x \sin x\,dx$.

Put $u = \sin x$, $\frac{dv}{dx} = e^x$.

Then $$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx.$$

To determine $\int e^x \cos x\, dx$, put $u = \cos x$, $\frac{dv}{dx} = e^x$, giving

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$$

whence $$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

or $$2\int e^x \sin x\, dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x\, dx = \frac{e^x(\sin x - \cos x)}{2}$$

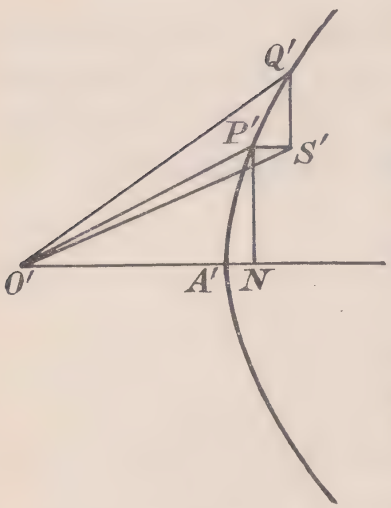


Fig. 4c.

The Hyperbolic Functions.

A good example of integration methods is afforded by the determination of the hyperbolic functions from the definitions given in Chapter I. Fig. 4A is here repeated, with slight modifications, as Fig. 4C. If $ON = x$ and $P'N = y$, $P'S' = \delta x$, $S'Q' = \delta y$ and $P'Q' = \delta s$ and angle $P'O'Q' = \delta\phi$. The first definition of $\delta\phi$ in hyperbolic radians is $\delta\phi = \frac{\delta s}{r}$, where $r = O'P'$.

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$
$$x^2 = y^2 + a^2$$
$$2x\, dx = 2y\, dy\,; \quad r^2 = x^2 + y^2$$

$$\delta y = \frac{x}{y}\delta x. \quad \text{Hence } (\delta s)^2 = (\delta x)^2\left(1 + \frac{x^2}{y^2}\right)$$

or $$\delta s = \frac{\delta x}{y} r$$

or $$\delta\phi = \frac{\delta s}{r} = \frac{\delta x}{y}$$

The second definition of $\delta\phi$ in hyperbolic radians is

$$\delta\phi = \frac{2 \times \text{area } O'P'Q'}{a^2}$$

But 2 area $O'P'Q'$

$$= 2 \text{ area } O'S'Q' - 2 \text{ area } O'P'S' - 2 \text{ area } P'Q'S'$$
$$= (x + \delta x)\delta y - y\,\delta x - \delta x\,\delta y$$
$$= x\,\delta y - y\,\delta x$$

$$\delta\phi = \frac{x\,\delta y - y\,\delta x}{a^2} = \frac{\frac{x^2}{y}\delta x - y\,\delta x}{x^2 - y^2} = \frac{\delta x}{y}$$

The two definitions therefore give the same value for $\delta\phi$.

$$\delta\phi = \frac{\delta x}{y} = \frac{\delta x}{\sqrt{x^2 - a^2}}$$

Hence $$\phi = \int_a^x \frac{dx}{\sqrt{x^2 - a^2}} = \Big[\log \frac{x + \sqrt{x^2 - a^2}}{a}\Big]_a^x$$
$$= \log \frac{x + \sqrt{x^2 - a^2}}{a}$$

Hence $$e^{\phi} = \frac{x + \sqrt{x^2 - a^2}}{a}$$

but $$\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)\left(\frac{x - \sqrt{x^2 - a^2}}{a}\right) = 1$$

and so $$e^{-\phi} = \frac{x - \sqrt{x^2 - a^2}}{a}$$

so that $$\frac{x}{a} = \frac{e^{\phi} + e^{-\phi}}{2} \text{ and } \frac{\sqrt{x^2 - a^2}}{a} = \frac{\varepsilon^{\phi} - \varepsilon^{-\phi}}{2}$$

But by definition $\cosh\phi = \frac{x}{a}$ and $\sinh\phi = \frac{y}{a} = \frac{\sqrt{x^2 - a^2}}{a}$

and so $\cosh\phi = \frac{e^{\phi} + e^{-\phi}}{2}$ and $\sinh\phi = \frac{e^{\phi} - e^{-\phi}}{2}$.

CHAPTER V

A BRIEF SURVEY OF FUNDAMENTAL ELECTROSTATIC, MAGNETIC AND ELECTRO-DYNAMIC RELATIONS

ELECTROSTATICS

Forces between Electric Charges.

The force in dynes between two point electrostatic charges q_1 and q_2 separated by a distance r cm. in a medium of *permittivity* κ (or *dielectric coefficient*, or *specific inductive capacity*) is

$$F = \frac{q_1 q_2}{\kappa r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The unit electrostatic charge is defined in terms of this equation, κ being unity for air or, more strictly, a vacuum.

Electric Field ; Lines of Force.

The vicinity of charges within which electric forces are experienced is the *electric field*. If a unit positive charge be placed at a point in the field the force acting upon it is termed the *electric force* or *electric intensity* or *strength of the field* at the point ; it is assumed that the disposition of all other charges is unaffected by the introduction of the unit charge. Electric force is a vector quantity. Faraday mapped out the electric field by means of *lines of force* so drawn that the tangent to the line of force at any point is the direction of the electric force. It will be seen later that the idea may be extended, through *tubes of force*, to express not only the direction of the field but also its strength.

Electric Potential.

The electric potential V at any point P in the electric field is the work which must be done in bringing unit positive

charge from the boundary of the field to P.* If two points P and Q on a line of force be separated by a small distance δs, the electric force being ε at P, then the work done by the electric force as the unit charge moves from P to $Q \doteqdot \varepsilon\, \delta s$. This is equal to the fall of potential between P and Q, $-\delta V$. It therefore follows that

$$\varepsilon = \underset{\delta s \to 0}{\text{Lt}} -\frac{\delta V}{\delta s} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The electric force is thus equal to the space rate of fall of potential, or to the *potential gradient.*

Electric Potential due to a Point Charge.

At a distance r from a charge q the electric force is $\varepsilon = \frac{q}{\kappa r^2}$. The work done in bringing unit positive charge from $r + \delta r$ to r is $\frac{q}{\kappa r^2}\delta r$. Hence the potential V at r is $\int_r^\infty \frac{q}{\kappa r^2} dr$ or

$$V = \frac{q}{\kappa r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If the field be due to a number of point charges then

$$V = \Sigma \frac{q}{\kappa r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Equipotential Surfaces.

If a surface be drawn everywhere normal to the lines of force no work is done in moving a small charge over the surface, there being no component of the electric force in a tangential direction. V is therefore constant and such a surface is called an *equipotential surface.*

Gauss's Theorem.

Consider a charge $+q$ in air. If a spherical surface of radius r be described with the charge as centre, the electric force is everywhere normal to the surface and is directed

* It is assumed here that the field possesses a potential—that the work done in bringing unit positive charge from the boundary of the field to any particular point is characteristic of the point and is independent of the path. This is treated more particularly later (p. 61).

outwards. The value of this electric force is q/r^2. The area of the sphere is $4\pi r^2$. If the value of the normal force be multiplied by the area over which it acts, namely the surface of the sphere, the product is termed the " total normal force." Its magnitude is $4\pi q$, independent of r and the same for every enclosing sphere. It is shown immediately below that the total normal force always has this value whatever the shape of the enclosing surface.

No attempt need be made to assign any particular physical significance to this quantity, the total normal force. It has the dimensions of force per unit charge multiplied by area. Its importance arises from the fact that it defines the total charge enclosed by the surface.

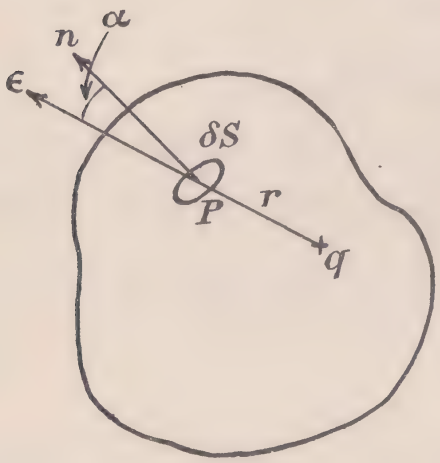


FIG. 8.

Considering the problem more generally the charge q, in air, is imagined completely enclosed by a continuous surface (Fig. 8). At a point P on the surface, distant r from q the outward drawn normal n makes an angle α with the electric force ε. The outward normal force on a small area δS enclosing P is $\varepsilon \cos\alpha\, \delta S = \frac{q}{r^2} \cos\alpha\, \delta S$ (since $\varepsilon = q/r^2$). The solid angle subtended at q by the elementary area δS is $\delta\omega = \frac{\delta S \,.\, \cos\alpha.}{r^2}$

Therefore the outward normal force over the area δS is $q\delta\omega$. It follows that the total normal force over the whole surface $= q\Sigma\delta\omega = 4\pi q$. If the surface contains several charges then the total normal force is equal to the sum of the total normal forces of the separate charges or $4\pi\Sigma q$. This is Gauss's theorem.

Digression on Partial Differentiation.

Partial differentiation will later be treated at some length, but it is here necessary to introduce the idea in order to establish two important extensions of Gauss's theorem which have wide application in electrical theory. In general, a phenomenon is dependent not only upon one variable but upon several. For

instance, the volume v of a quantity of gas depends upon its temperature θ and upon its pressure p. If θ and p change, the consequent change of volume may be expressed as the sum of (a) the change caused by change of θ, p remaining constant, and (b) the change caused by change of p, θ remaining constant. The rate of change of v with respect to θ, p being constant, is denoted by $\frac{\partial v}{\partial \theta}$ — the *partial differential coefficient* of v with respect to θ. Similarly the rate of change of v with respect to p, θ being constant, is $\frac{\partial v}{\partial p}$. For small changes of temperature and pressure, $\delta\theta$ and δp, the consequent small change in volume δv may be approximately expressed as

$$\delta v \doteqdot \frac{\partial v}{\partial \theta}\delta\theta + \frac{\partial v}{\partial p}\delta p \quad . \quad . \quad . \quad . \quad . \quad (44)$$

This relation becomes exact when $\delta\theta$ and δp become vanishingly small when it is expressed as

$$dv = \frac{\partial v}{\partial \theta}d\theta + \frac{\partial v}{\partial p}dp \quad . \quad . \quad . \quad . \quad . \quad (45)$$

A more pertinent example in the present instance is provided by the variation of potential and intensity in an electric field. If in the field Cartesian co-ordinates be placed the potential V and the electric force ε will in general vary with x, y and z. The magnitude and direction of the electric force ε at any point P may be defined by its components X, Y and Z along the co-ordinate axes of x, y and z. Then $X = -\frac{\partial V}{\partial x}$, $Y = -\frac{\partial V}{\partial y}$ and $Z = -\frac{\partial V}{\partial z}$. The rate of change of X along x is similarly $\frac{\partial X}{\partial x} = -\frac{\partial^2 V}{\partial x^2}$; of Y along y, $\frac{\partial Y}{\partial y} = -\frac{\partial^2 V}{\partial y^2}$; and of Z along z, $\frac{\partial Z}{\partial z} = -\frac{\partial^2 V}{\partial z^2}$.

Poisson's Equation.

If Gauss's theorem be applied to the elementary parallelepiped of Fig. 9, it is seen that the outward normal force contributed by the face A is $-X\,dy\,dz$ and by the face B,

$\left(X + \frac{\partial X}{\partial x}dx\right)dy\,dz$. The nett force contributed by these two faces is therefore

$$\frac{\partial X}{\partial x}dx\,dy\,dz$$

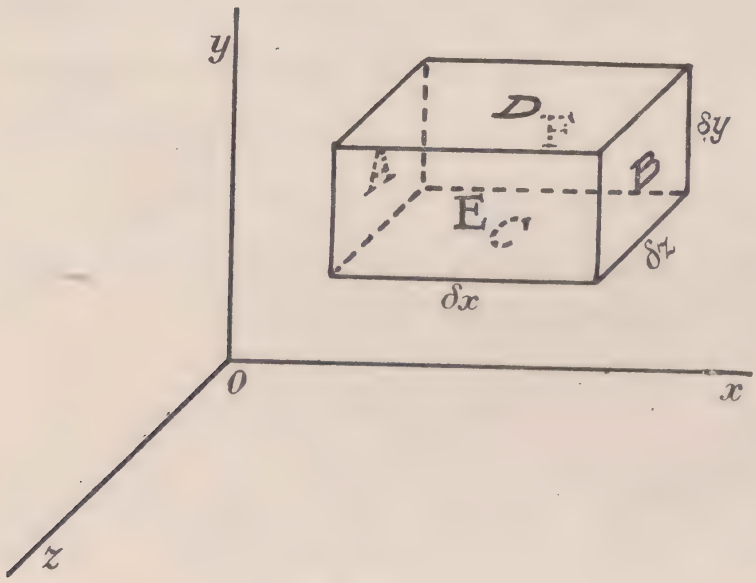


Fig. 9.

Proceeding similarly with the other faces in pairs it is seen that the total outward normal force is

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)dx\,dy\,dz$$

If at the point considered there is a volume distribution of electricity of density ρ then, by Gauss's theorem,

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)dx\,dy\,dz = 4\pi\rho\,dx\,dy\,dz$$

or

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 4\pi\rho \quad . \quad . \quad . \quad . \quad (46)$$

$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$ is said to be the *divergence* of the vector X, Y, Z and is written div (X, Y, Z) or div ε. Equation 46 is more usually expressed in terms of the scalar quantity V. It then becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0 \quad . \quad . \quad . \quad (47)$$

and is commonly written $\nabla^2 V + 4\pi\rho = 0 \quad . \quad . \quad . \quad . \quad (48)$

(∇ is pronounced del). This is Poisson's equation.

Laplace's Equation.

In free space $\rho = 0$ and Poisson's equation reduces to Laplace's equation

$$* \operatorname{div} \varepsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

or

$$\nabla^2 V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This equation has a wide application in potential theory, not only in electricity but also in acoustics and other branches of physics.

Tubes of Force.

An elementary tube of force may be considered as emanating from any small area $\delta S''$ of a positively electrified body, its walls being formed by the consecutive lines of force springing from the periphery of $\delta S''$ (Fig. 10). At points P and Q,

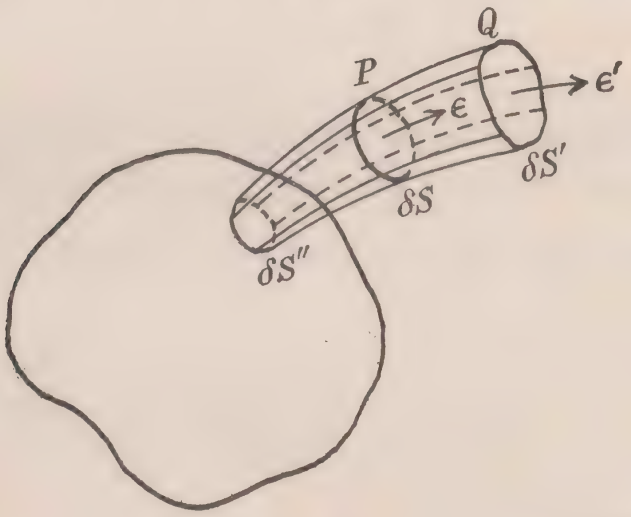


Fig. 10.

where the electric force is ε and ε' respectively the areas of intersection with the equipotential surfaces are δS and $\delta S'$. Considering the closed surface formed by the tube wall and δS and $\delta S'$, the total outward normal electric force is zero, since it contains no charge. There is no normal component at the tube wall and hence

$$\varepsilon \, \delta S = \varepsilon' \, \delta S' \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Surface Density of Charge.

The electric force ε_s at the surface of a conductor is closely connected with the surface density of charge, σ. A small

* A vector whose divergence is zero within a region is said to be solenoidal in that region. In free space electric force is a solenoidal vector.

closed surface will be considered which consists of a short length of a tube of force closed at its outer end by an equipotential; the surface is completed in any manner within the conductor. This closed surface envelops a small area $\delta S''$ of the conducting surface (Fig. 11). The total outward normal force is $\varepsilon_s \, \delta S''$. This must be equal to $4\pi q$ or $4\pi\sigma \, \delta S''$. Whence

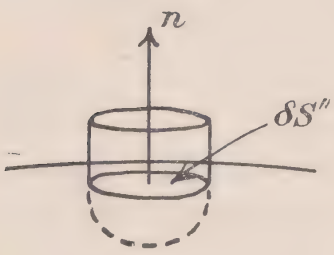


Fig. 11.

$$\sigma = \frac{\varepsilon_s}{4\pi} = -\frac{1}{4\pi}\frac{\partial V}{\partial n}$$

where n is the outward drawn normal to the conducting surface. More generally, if the medium is not air

$$\sigma = \frac{\kappa\varepsilon_s}{4\pi} = -\frac{\kappa}{4\pi}\frac{\partial V}{\partial n} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Equality of Magnitude of Charges at the Ends of a Tube.

An elementary tube of force starts at a positive charge $\sigma \, \delta S''$ or $\frac{\varepsilon_s \, \delta S''}{4\pi}$. But throughout the tube the product $\varepsilon \, \delta S$ is constant (equation 51); hence the tube must end at a negative charge, equal in magnitude to that from which it started. A tube which starts at a unit positive charge and finishes at a unit negative charge is a unit, or Faraday, tube.

Electric Displacement.

The elementary area $\delta S''$ may be considered divided into sections from each of which a Faraday tube starts. The number of such tubes is $\sigma \, \delta S''$ (the number may, of course, be fractional). At any point along the main tube the number of Faraday tubes per sq. cm. (measured perpendicular to the tubes) is

$$D = \frac{\sigma \, \delta S''}{\delta S} = \frac{\varepsilon_s \, \delta S''}{4\pi \, \delta S} = \frac{\varepsilon \, \delta S}{4\pi \, \delta S} = \frac{\varepsilon}{4\pi} \text{ giving}$$

$$\varepsilon = 4\pi D$$

If the medium is not air

$$\varepsilon = \frac{4\pi D}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

D is the *displacement* (or *polarisation* or *electric flux density*).

Capacitance.

The map of the electric field is determined by the disposition and relative magnitudes of the charges producing it. If the disposition of the charges be unaltered while they are all increased in a definite ratio the direction of the field will be unaltered and its intensity everywhere will be increased in the same ratio as the charges. There will consequently be the same proportional increase in the potential at all points in the field and in the difference of potential between particular conductors. If the field is due to charges $+q$ and $-q$ on two conductors, well removed from other influences, the difference of potential between them will be proportional to q. If V be this difference of potential then

$$q = CV \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where C is a constant and is termed the *capacitance* of the system. When the conductors are deliberately disposed so that the charges $+q$ and $-q$ are large for a given difference of potential then the two conductors form a *condenser*.

When a single conductor is well removed from others its capacitance is given by its charge divided by its potential.

Replacement of an Equipotential Surface by a Conducting Surface. Capacitance of a Sphere.

It is clear that a conducting surface, if given an appropriate charge, may replace an equipotential surface without in any way disturbing the electrical conditions exterior to itself. The appropriate charge is defined by the condition that the total electric flux, in Faraday tubes, issuing from the surface must be the same as that which issued from the equipotential surface. Since the tubes are continuous through the equipotential surface the nett charge on the replacing conducting surface is zero if any contained charges are unchanged. If the contained charges are annulled an equal total charge must be given to the conducting surface. The same condition is given by defining the potential of the conductor as that of the equipotential surface which it replaces.

A particularly simple application of this principle is provided in the case of a point charge q well removed from other charges or conducting bodies. The equipotential surfaces are spheres and, as far as external conditions are concerned, any one of

these spherical equipotential surfaces may be replaced by a spherical conducting surface having a charge q—the original point charge being annulled. This gives the important result that the electric force at any point external to a charged sphere is the same as it would be if the charge were concentrated at the centre of the sphere. For a point charge q the potential at a distance r is $V = q/\kappa r$ (equation 42). This is therefore the potential of an isolated charged sphere of radius r. The capacitance of the sphere is therefore κr.

Capacitance of Two Concentric Spheres.

If the inner of two concentric conducting spheres be given a charge $+ q$ it is evident that the inner surface of the outer sphere must have a charge $- q$ (p. 46). If this is the total charge of the larger sphere, no tubes of force can radiate from its outer surface and therefore its potential must be zero. The difference of potential between the two spheres is the actual potential of the inner sphere. At any radius r, between the spheres, the electric force

$$\varepsilon = -\frac{dV}{dr} = \frac{q}{\kappa r^2}$$

whence

$$V = \frac{q}{\kappa r} + c$$

If the inner sphere is of radius a and the outer sphere of radius b, the constant of integration is given by $V = 0$ where $r = b$ or $0 = \frac{q}{\kappa b} + c$. Hence the difference of potential of the two spheres is

$$\frac{q}{\kappa a} - \frac{q}{\kappa b} = \frac{q}{\kappa}\left(\frac{b-a}{ab}\right)$$

Hence the capacitance of the condenser so formed is

$$C = \kappa \frac{ab}{b-a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Capacitance of Parallel Plate Condenser.

The result just given for two concentric spheres may be written as

$$C = \kappa \frac{a(a+d)}{d}$$

where d is the separation of the two spheres. The capacitance per unit area of the inner sphere is thus

$$C' = \frac{\kappa}{4\pi d}\left(1 + \frac{d}{a}\right)$$

If now a be imagined indefinitely increased then a parallel plate condenser is obtained, each plate being of unit area. The capacitance of a condenser consisting of two parallel plates, each of area A separated by a distance d, is thus

$$C = \frac{\kappa A}{4\pi d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Potential Energy of a Condenser.

If a condenser of capacitance C has its charge increased by transference of a small charge δq from the negative plate to the positive the work done is $V\delta q = \frac{q}{C}\delta q$. The total energy in the condenser is thus

$$W = \int_0^q \frac{q}{C} dq = \frac{q^2}{2C} = \tfrac{1}{2}CV^2 = \tfrac{1}{2}qV \quad . \quad . \quad . \quad (57)$$

Energy of the Electric Field.

The energy of a condenser may be considered as being associated with the electric field. Per unit area of plate, charged to a density σ, the energy is, from equations 56 and 57

$$\frac{\sigma^2}{2C} = \frac{2\pi\sigma^2}{\kappa} d \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

If the charge is maintained constant the energy is proportional to the distance between the plates, this energy being derived from the mechanical work done in separating the plates. As the plates are separated the field remains of the same intensity but is increased in volume. The force between the plates is independent of the distance between them so long as the distance is small compared with the size of the plates.

The energy of the field per c.c. is $\frac{2\pi\sigma^2}{\kappa}$. Since the Faraday tubes are parallel $D = \sigma$, but $D = \frac{\kappa\varepsilon}{4\pi}$. Hence the energy per c.c. is

$$W_1 = \frac{2\pi}{\kappa} \cdot \frac{\kappa^2\varepsilon^2}{16\pi^2} = \frac{\kappa\varepsilon^2}{8\pi} \quad . \quad . \quad . \quad . \quad (59)$$

MAGNETISM

Force between Two Magnetic Poles ; Magnetic Moment of a Magnet.

The force between two magnetic poles separated by a distance r in air or a non-magnetic medium is

$$F = \frac{m_1 m_2}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

The unit magnetic pole is defined in terms of this equation.

The simplest magnet consists of two poles of strength $+ m$ and $- m$ separated by a distance l. The product ml is the *magnetic moment* M.

The total magnetic charge on any magnet is zero. The conception of a unit pole is useful, but care must be exercised in deducing relations which assume its actual existence since it can only exist in association with an equal pole of the opposite sign.

Magnetic Field ; Lines of Force.

As in the electrical case the magnetic field may be mapped by lines of force and tubes of force. The strength of the magnetic field at any point, or *magnetic force*, is defined as the force acting upon unit magnetic pole if placed at the point. As in the electrostatic case, Gauss's theorem shows that the total flux of force from a magnetic pole of strength m is $4\pi m$. Tubes of force are imagined to exist so that their density at any point in space is equal to the magnetic force. There are thus 4π tubes of force radiating from a unit magnetic pole.

Magnetic Potential.

The potential at any point of a magnetic field is the work which must be done in bringing unit pole from the boundary of the field to the point in question. If at any point the magnetic force is H and the potential V then

$$H = -\frac{dV}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where s is measured along a line of force.

Magnetic Potential due to a Short Bar Magnet.

Due to an individual pole of strength m, the potential at

a point P, distant x from the pole, must be $\frac{m}{x}$ (equation 42). If a short bar magnet of length l be considered (Fig. 12), the potential at the point P defined by the co-ordinates r and θ must be

$$V = \frac{m}{NP} - \frac{m}{SP}$$

But $$NP = \sqrt{r^2 + \frac{l^2}{4} - rl\cos\theta}$$

and $$SP = \sqrt{r^2 + \frac{l^2}{4} + rl\cos\theta.}$$

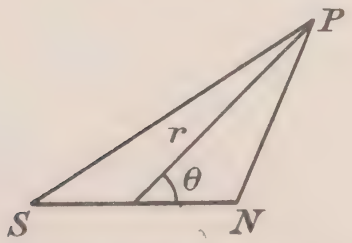


Fig. 12.

If it be imagined that l is made indefinitely short, the product ml remaining constant and equal to M, the potential V may be written as

$$V = \frac{M}{l}\left(\frac{1}{\sqrt{r^2 - rl\cos\theta}} - \frac{1}{\sqrt{r^2 + rl\cos\theta}}\right)$$

$$= \frac{M}{rl}\left[\left(1 + \frac{l}{2r}\cos\theta\right) - \left(1 - \frac{l}{2r}\cos\theta\right)\right]$$

or $$V = \frac{M\cos\theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Potential due to a Magnetic Shell.

A magnetic shell consists of an indefinitely thin sheet of iron magnetised perpendicularly to its surface. The magnetic moment per unit area is its strength g. At a point P, distant r from an elementary area δS on the shell (Fig. 13) the potential due to δS is

$$\delta V = \frac{g\delta S\cos\theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

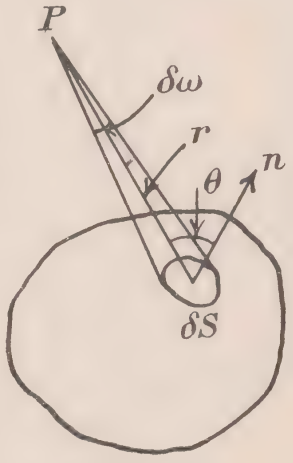


Fig. 13.

where θ is the angle between r and the normal n to the surface.

But $\frac{\delta S \cos\theta}{r^2} = \delta\omega$, the small solid angle subtended by δS at P.

Hence $$V = g\Omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where Ω is the solid angle subtended by the whole shell at P.

Intensity of Magnetisation.

If an iron bar of length l and cross-sectional area a be uniformly magnetised along its length, the pole strength at its ends being J per sq. cm., J is called the intensity of magnetisation. The magnetic moment of such a magnet is $Jal = Jv$ where v is the volume. Any elementary volume of a generally magnetised piece of iron may be considered as being uniformly magnetised. The intensity of magnetisation of such an elementary volume may be defined as its elementary magnetic moment divided by its volume or $J = \frac{\delta M}{\delta v}$.

Magnetic Field between two Equally and Uniformly Magnetised Poles.

Two long equally and uniformly magnetised iron bars are arranged with their axes in line so that two dissimilar poles are separated by a distance which is very small compared with the diameter of either (Fig. 14A). It is required to determine the magnitude of the magnetic force in the gap. A point P distant p from the face of one bar is considered. Each element of a ring of width δr (Fig. 14B) exerts a magnetic force at P which is

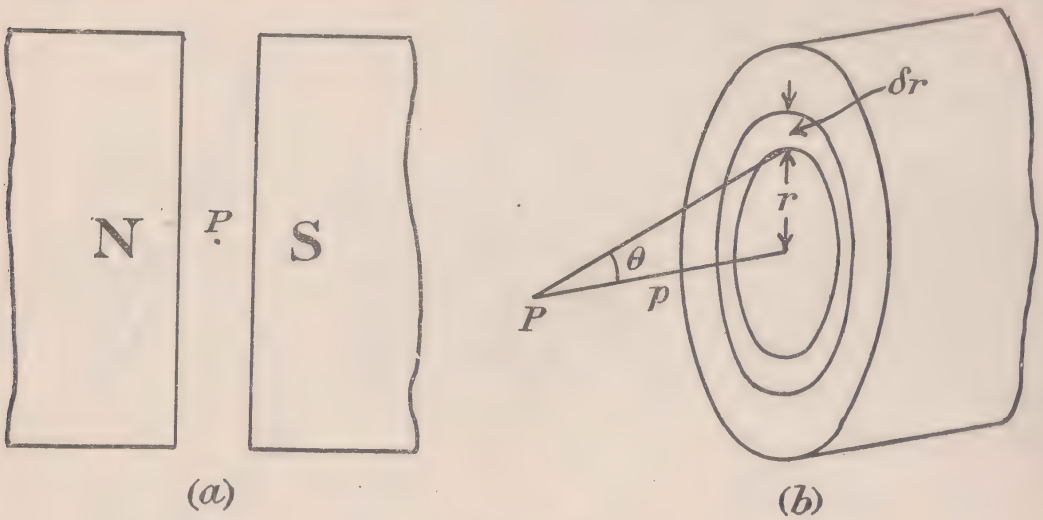


FIG. 14.

inclined at an angle θ to the normal. The force due to an elementary area enclosed between two radii separated by an angle $\delta\phi$ is

$$\frac{Jr\delta\phi\delta r}{p^2 + r^2}$$

The resultant force due to this elementary area normal to the face is

$$Jp\delta\phi\delta r\frac{r}{(p^2 + r^2)^{\frac{3}{2}}}$$

The resultant force due to the ring is

$$2\pi Jp\delta r\frac{r}{(p^2 + r^2)^{\frac{3}{2}}}$$

The total force due to the whole face is

$$2\pi Jp\int_0^\infty \frac{r}{(p^2 + r^2)^{\frac{3}{2}}}dr = -2\pi Jp\Big[\frac{1}{\sqrt{p^2 + r^2}}\Big]_0^\infty = 2\pi J$$

(the upper limit can be made infinite since it is assumed that the maximum value of r is very great compared with p).

There will be an equal force of $2\pi J$ due to the other pole and the total magnetic force in the gap is $4\pi J$. This result might have been deduced more simply. The tubes of force proceed from the N pole to the S pole. When the poles are close together these tubes are parallel and their density is $4\pi J$ since 4π tubes leave each unit N, or positive, pole. This gives the intensity of the magnetic field (p. 50).

ELECTRODYNAMICS

Magnetic Field produced by an Electric Current. Force upon a Conductor in a Magnetic Field.

When electricity is in motion magnetic forces are experienced in its neighbourhood. In the case of a wire conveying a current the strength of the magnetic field falls off as one retreats from the wire. The strength of the field increases with increase in the magnitude of the current and with increase in the length of the wire carrying the current. It is this magnetic effect which is used to define the magnitude of an electric current.

The C.G.S. electromagnetic unit of current is such that when flowing along 1 cm. of wire bent into the arc of a circle of 1 cm.

radius, it exerts a force of 1 dyne on a unit magnetic pole placed at the centre of the circle of which the arc forms a part. Consequently if this current flows through the complete circle of 1 cm. radius the strength of the magnetic field at the centre of the circle is 2π. The ampere is one-tenth of this C.G.S. electromagnetic unit of current. The direction of the magnetic force is indicated in Fig. 15. If the current is clockwise the magnetic force is perpendicular to the plane of the paper and *into* the paper.

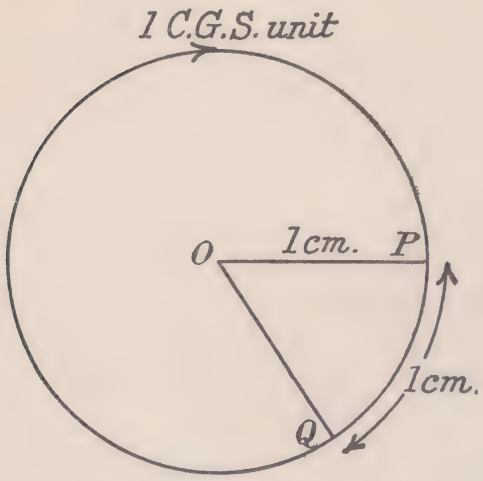


FIG. 15.

Considering the cm. PQ of wire conveying unit current it is exerting a force of 1 dyne upon the unit magnetic pole placed at O. It therefore follows that there is an equal and opposite force acting upon the length PQ, urging it perpendicularly out of the paper. But PQ is in a magnetic field of unit strength directed perpendicularly to its length, due to the magnetic pole. That is to say a conductor conveying a current in a magnetic field is acted upon by a force perpendicular to the field and to the conductor. If the unit pole be increased to H, producing a field of H at a distance of 1 cm., and PQ be increased in length to l cm., and the current be increased to I_a C.G.S. units, then the mechanical force acting upon PQ is

$$F = lHI_a \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad (65)$$

If the current is I amperes then

$$F = \frac{1}{10} lHI \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Magnetic Field due to any Element of a Circuit.

The strength of the magnetic field at a distance r from a unit magnetic pole is $1/r^2$. If at this point there is an element δl of a circuit conveying a current I_a C.G.S. units, the angle between δl and r being θ, then the force exerted by the pole upon the circuit element is

$$\delta F = I_a \frac{\delta l}{r^2} \sin\theta \quad . \quad . \quad . \quad . \quad . \quad (67)$$

This must therefore be the force upon the pole, or the strength of the magnetic field due to the circuit element at the position of the pole.

Magnetic Moment of an Electric Circuit.

If a current circulates in a single turn of conductor (which need not be in one plane) this circuit will, in a magnetic field, tend to set itself in a definite direction. Its magnetic axis is then coincident with the direction of the field. If the circuit be turned so that its axis is perpendicular to the field it will be acted upon by a moment tending to turn it back again. If the field is of unit strength the value of this moment is the magnetic moment of the circuit. Its value is $I_a a$ where a is the projected area of the circuit in the direction of its axis (its actual area if all in one plane). This is shown simply as follows. In Fig. 16 the magnetic axis of the circuit is imagined to be perpendicular to the plane of the paper, the direction being inwards as indicated by the barb of the arrow. The lines of magnetic force due to the external field are parallel to the plane of the paper and directed downwards as shown. Two planes separated by a distance δx are shown by their traces. These planes cut the circuit and define two short lengths δs and $\delta s'$. The length δs is urged into the paper and the component of the force normal to the plane is $H \times$ (projection of δs on to the plane of the paper) $\times \sin \theta \times I_a = HI_a \, \delta x$. If H is unity the component force is $I_a \, \delta x$. Similarly the component force acting upon $\delta s'$ is $I_a \, \delta x$ normally outwards from the paper. The moment operating upon these two elements is $I_a y \, \delta x$. But $y \, \delta x = \delta a$, the element of the projected area on to the plane of the paper between the two planes defining δx. It therefore follows that the total moment acting upon the circuit is $I_a \Sigma \delta a$ or

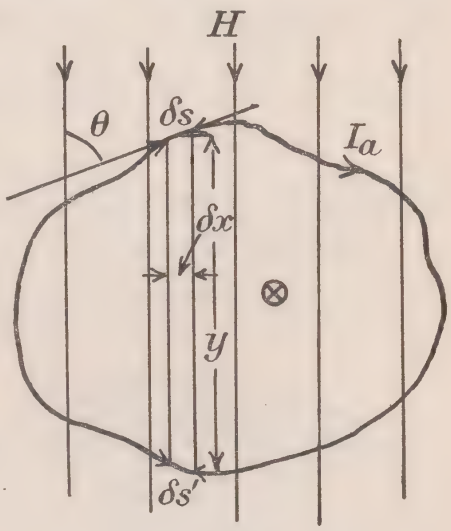


Fig. 16.

$$M = I_a a \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

By the same argument it follows that if the circuit be tipped

in any way so that its projected area viewed from a direction perpendicular to the field is a', then the component magnetic moment having for axis the line of view is

$$M' = I_a a' \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

If the tipping be carried to the point that along the line of view the current appears to be going anticlockwise the component of the magnetic moment is negative. This may be considered as being due to the projected area being negative. This is seen to be a consistent interpretation if one considers the case of a plane area viewed at an angle θ to the normal. The apparent, or projected, area is $a \cos \theta$ which becomes negative when θ exceeds $\frac{\pi}{2}$ but is less than π.

Magnetic Potential due to an Electric Current in a Closed Circuit.

A closed circuit carries a current I_a C.G.S. units. It is required to find the magnetic potential at a point P in the field. The area enclosed by the circuit may be imagined to be divided up by a fine conducting network forming squares, or approximate squares, round each of which a current I_a circulates in the same sense. It is seen from Fig. 17 that, except in those elements of conductor forming the boundary circuit, the current in each element is zero, since equal and opposite currents are carried by the element. This network may therefore be substituted for the actual circuit.

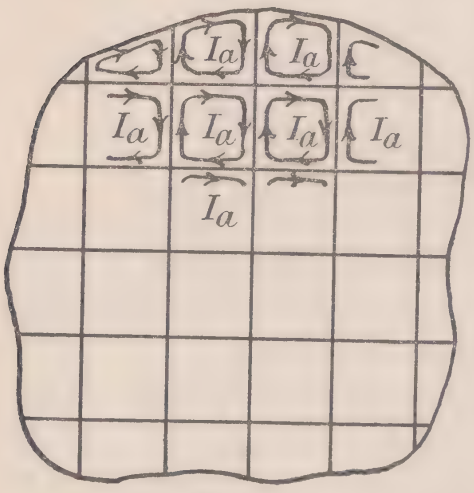


FIG. 17.

Due to a unit magnetic pole placed at P the forces operating upon an elementary area δS of the network are the same as those operating upon an infinitesimal magnet of normal moment $I_a \delta S$ (equation 68). It therefore follows that the field produced at P by the element δS is the same as that produced by an element of a magnetic shell of area δS and of strength I_a. Hence the circuit is equivalent to a magnetic shell, having the

circuit as boundary, of strength $g = I_a$. The potential at a point P is therefore

$$V = I_a \Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where Ω is the solid angle subtended by the circuit at P.

Force on a Moving Charge in a Magnetic Field.

The magnetic force at the centre of the circle of Fig. 15 is produced by the moving charges constituting the electric current. These charges are moving with a statistical average velocity u. If the charge in motion on a centimetre length is q, the quantity passing any point per second is qu. This is the current. The intensity of the field produced at the centre of the circle by the centimetre length is qu. If the density of charge be increased n times and move with the same velocity the current is increased to nqu and the intensity of the field produced at the centre of the circle by $\frac{1}{n}$th cm. of the circuit is still qu. The factor n may be increased indefinitely when it is seen that a charge q moving with a velocity u produces an intensity of magnetic field of qu C.G.S. units at a distance of 1 cm. measured perpendicularly to its path. If the charge be in a field of intensity H, directed perpendicularly to the path, the force exerted upon it is

$$F = Hqu \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Electromotive Force generated in a Conductor moving in a Magnetic Field.*

A conductor AB of length l is imagined to move, with a velocity u, perpendicularly to its length in a field of intensity H directed perpendicularly to l and u (Fig. 18). Charges on this conductor are acted upon by forces tending to move them along the wire and they redistribute themselves so that equilibrium is established under the action of the forces and the

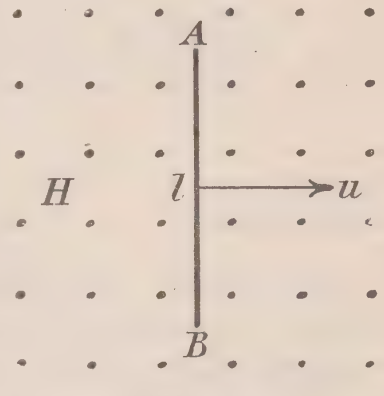


FIG. 18.

* This method of obtaining the e.m.f. generated in a conductor is due to T. B. Vinycomb. The usual method of " proving " the result given in equation 72 involves a conductor sliding upon a pair of rails so that the flux through the circuit increases as the conductor moves. It is shown later that in such a conductor an e.m.f. is generated even when $H = 0$.

variation of potential along the wire. Any charge q which has moved a distance x has had mechanical work done upon it equal to $Hqux$. But this is equal to $V_x q$, where V_x is the difference of potential over the length x. Hence $V_x = xHu$ or the e.m.f. generated over the whole length l is

$$E = lHu \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Work done in changing the Magnetic Flux linking a Circuit.

A simple fixed circuit is considered carrying a current I_a supplied by a battery B (Fig. 19). Of the flux issuing from the pole of a magnet an amount ϕ links the circuit; the polarity is imagined to be such that the magnet is repelled. On approaching the magnet towards the circuit the flux linkage is increased by lines cutting the circuit. An e.m.f. is generated equal to $d\phi/dt$; mechanical work is done and electrical energy generated. The e.m.f. of the battery must be reduced to maintain the current at the constant value I_a. The rate at which work is done is equal to $I_a \times$ e.m.f. generated $= I_a \frac{d\phi}{dt}$. The total work done $= \int I_a \frac{d\phi}{dt}\, dt = I_a \times$ change in flux linkages.

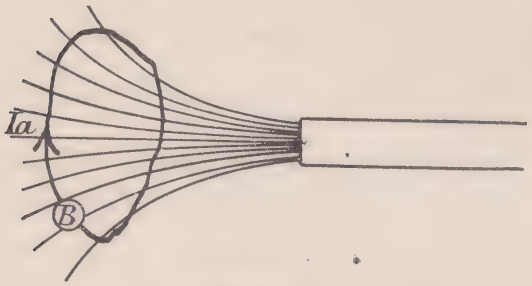


FIG. 19.

Inductance.

Associated with any circuit carrying an electric current there is a magnetic flux. If the current changes the flux also changes and an e.m.f. $d\phi/dt$ is generated in the circuit. The direction of this e.m.f. must be such as to oppose the change of current. In a circuit which does not contain a magnetic substance the flux is always proportional to the current and may be written as

$$\phi = LI_a \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

L is the *inductance* * of the circuit. Equation 73 may be differentiated giving

$$\varepsilon = -\frac{d\phi}{dt} = -L\frac{dI_a}{dt} \quad . \quad . \quad . \quad . \quad . \quad (74)$$

If another circuit be in the proximity of the circuit considered then the change of current in the first circuit produces a change of flux and an e.m.f. in the second. This secondary e.m.f. may be written

$$\varepsilon_2 = -M\frac{dI_a}{dt} \quad . \quad . \quad . \quad . \quad . \quad (75)$$

M is the *mutual inductance* of the two circuits.

Energy associated with a Current in an Inductance.

If an e.m.f. E be applied to a resistanceless circuit of inductance L current grows at the rate

$$\frac{dI_a}{dt} = \frac{E}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

If E be constant the rate at which energy is being given to the circuit at any instant is EI_a, and the total energy given in a time t is $E\int_0^t I_a\,dt$. From equation 76

$$I_a = \int_0^t \frac{E}{L}dt = \frac{E}{L}t$$

Hence the energy of the circuit after a time t is

$$W = \frac{E^2}{L}\int_0^t t\,dt = \frac{E^2t^2}{2L} = \tfrac{1}{2}LI_a^2 \quad . \quad . \quad . \quad (77)$$

Vector Fields ; Potential.

Two examples of vector fields have so far been treated—the electric field produced by stationary and unvarying electric charges and the magnetic field produced by a system of magnets or currents. In the case of the electric field the vector electric intensity ε, varying from point to point, can be expressed in terms of a scalar function—the potential. The case of the magnetic field is not quite so simple.

Potential is defined as a scalar point function whose gradient

* Strictly speaking it is inadmissible to talk of the inductance of an iron cored circuit. There is little objection when the limitation is clearly recognised ; unfortunately this is not always the case.

is a vector field. The meaning of this definition is clearly seen in the case of the electric field. It was indeed definitely so expressed in equation 41 :

$$\varepsilon = \underset{\delta s \to 0}{\text{Lt}} \; -\frac{\partial V}{\partial s}.$$

It is not the case, however, that every vector field possesses a potential. A simple criterion for the existence of a potential is that the *curl* of the field must be zero. The term *curl* has so far not been introduced ; it will be defined later. It is sufficient for present purposes to simplify the condition to requiring that the *line integral* of the vector round any closed curve in the field must be zero. The *line integral* of any vector α is defined as $\int \alpha \cos\theta \, ds$ (Fig. 20).

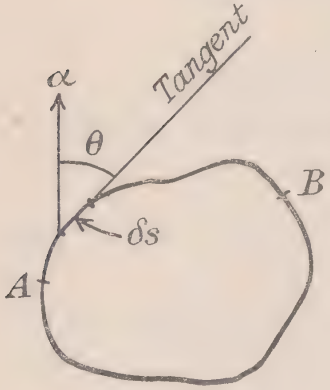


FIG. 20.

Its physical significance depends upon the nature of the vector. If α is a force then $\alpha \cos\theta$ is the component of the force in the direction of motion and the line integral between two points is the work done along that part of the path. If for instance an electric field is being considered and α is the force upon unit charge, or ε, then $\int_A^B \varepsilon \cos\theta \, ds$ is the work done in taking unit charge from A to B along the particular path and has been called the difference of potential between A and B. If the potential is a point function having a real meaning then the line integral must be independent of the path and must only depend upon the conditions at A and B. This is the case if the integral round a closed path is zero. For if the path A to B (Fig. 20) is traversed in a clockwise direction, the same line integral is obtained as in proceeding from A to B in an anticlockwise direction if the line integral from A and back to A is zero. An integral taken completely round a curve may be conveniently written as $\oint$. The condition that the vector α shall have a potential is

$$\oint \alpha \cos\theta \, ds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Line Integral in the Electric Field.

In dealing with the electric field produced by a number of fixed charges it has been assumed (pp. 40, 41) that a potential exists. This is simply proved. A single charge is first considered. If any closed path (not necessarily in one plane) be drawn in the field an element δs of the path may be considered as being intercepted between two concentric spheres of radii r and $r + \delta r$ having the charge as centre. Then

$$\delta r = \delta s \cos \theta$$

where θ is the angle between the radius and the tangent to the path at the point, and hence

$$\varepsilon \cos \theta \, \delta s = \varepsilon \, \delta r$$

But since the path is closed there must be another element intercepted between the two spheres which is traversed in the opposite direction (opposite meaning from $r + \delta r$ to r if the original movement were from r to $r + \delta r$) and at this second point of the path ε must have the same value, namely q/r^2. Hence the sum of these two elements of the line integral is zero. The whole path may similarly be divided into elementary pairs and the result is obtained

$$\oint \varepsilon \cos \theta \, ds = 0 \quad . \quad . \quad . \quad . \quad . \quad (79)$$

But the actual value of the electric force at a point is the sum of all the separate forces due to the separate charges and the line integral is the sum of the line integrals due to each charge separately. It therefore follows that for fixed unvarying charges, equation 79 is universally true. In bringing unit charge from the boundary of an electrostatic field to a particular point the work done is independent of the path and the point has a definite potential.

Potential in a Magnetic Field.

The case of a magnetic field produced by a current in a circuit is not so simple. On page 57 it has been shown that the magnetic potential at a point P is $V = I_a \Omega$ where Ω is the solid angle subtended by the circuit at the point P. If the argument be followed, it will be seen that under the conditions implicitly assumed, but not explicitly stated, this value for V is obtained whatever the path from the boundary of the field to

the point P. But under other conditions a different value may be assigned to V, for it is not true that the line integral of H round a closed curve is always zero. It often is, but actually it may possess a number of discrete values expressed as

$$\oint H \cos\theta \, ds = I_a \cdot 4\pi n \quad . \quad . \quad . \quad . \quad (80)$$

where n may have any *integral* value. The potential at a point P may then be written as

$$V = I_a(\Omega + 4\pi n) \quad . \quad . \quad . \quad . \quad . \quad (81)$$

(Note the analogy with plane angles where the inclination between two lines may be expressed as $\phi + 2\pi n$ where n may have any integral value. In this expression 2π is the angle subtended at the centre of a circle by the complete circumference. In equation 81, 4π is the angle subtended at the centre of a sphere by the complete surface.) According to the general definition given on page 59 to 60 the magnetic field might be said not to have a potential. It is, however, more useful to say that it has a many-valued potential.

The condition implicitly assumed in obtaining equation 70 was that in approaching the circuit from the boundary of the field the solid angle subtended by the circuit grew steadily from zero to Ω. Under these conditions

$$\int_{\infty}^{P} H \cos\theta \, ds = I_a\Omega$$

If the circuit be approached closer the angle increases to 2π when the plane (or average plane) of the circuit is reached and passes beyond 2π as the point passes through the circuit. Starting from any point P and passing through the circuit and back again to P outside the circuit the solid angle swept is 4π. Therefore

$$\oint H \cos\theta \, ds = I_a \cdot 4\pi \quad . \quad . \quad . \quad . \quad (82)$$

for any path linking the circuit once.

If the circuit be linked n times

$$^{n}\oint H \cos\theta \, ds = I_a \cdot 4\pi n \quad . \quad . \quad . \quad . \quad (83)$$

The line integral of the electric force round any closed path is the *electromotive force* (the term electromotive force is excusably

used in other senses, but when so used no confusion is likely to arise).

The line integral of the magnetic force round any closed path is the *magnetomotive force. In an electrostatic field the e.m.f. round any closed path is zero. In a magnetic field the m.m.f. is equal to* 4π *times the current linked.* (When dealing with a path partly or wholly through iron it is usual in reckoning the magnetomotive force to include only those currents deliberately supplied from some source, such as batteries or generators and not to include atomic circuits. These latter constitute the magnetisation of the iron and they are treated

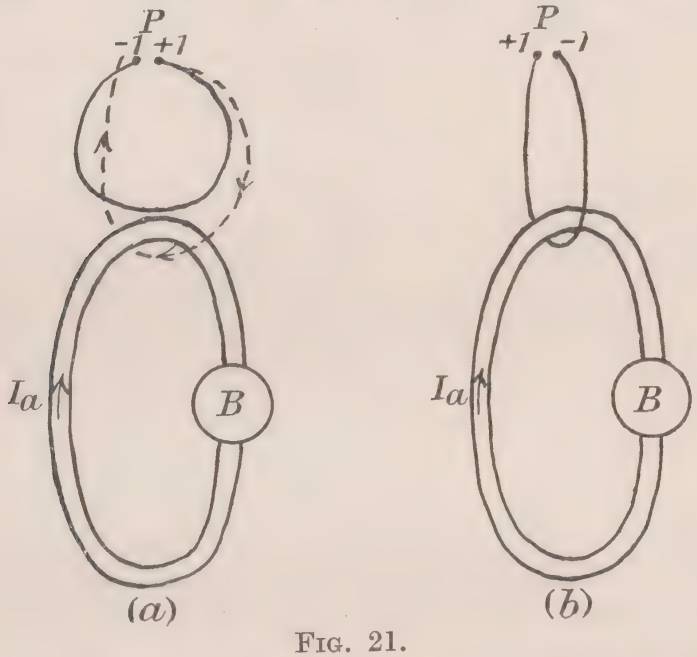


FIG. 21.

separately.) This particular result may be obtained in the following manner. A very flexible magnet is imagined having poles of $+1$ and -1 units at its ends. Passing through the magnet there is a magnetic flux 4π which issues from the positive pole to re-enter at the negative pole. The magnet is folded so that its two poles coincide at a point P in the neighbourhood of a circuit carrying a current I_a (Fig. 21A). The negative pole remaining fixed, the positive pole is swept through the path indicated until the poles are again coincident (Fig. 21B). The force exerted by the circuit upon the pole is H and the work done is $\oint H \cos\theta \, ds$, which is supplied to the circuit. But the work done is also $4\pi I_a$ (p. 58). $\oint H \cos\theta \, ds$

is the m.m.f. round the path which is thus equal to 4π times the current linked.

The fact that the magnetic potential is multivalued rarely causes any difficulty. Any ambiguity which may arise can in fact be resolved by imagining that the conductor forming any electric circuit is the boundary of a surface which must not be penetrated. Under such conditions the line integral round any closed curve becomes zero.

Curl of a Vector.

It has been seen that an electrostatic field possesses a potential. The line integral of the electric force from the boundary of the field is independent of the path, and the line integral round any closed path is zero. A magnetic field produced by a current in a circuit has a many-valued potential. The line integral round any closed path may be zero or $4\pi n I_a$, n being integral. If the circuit be linked n times, the line integral is independent of the path followed during the linking.

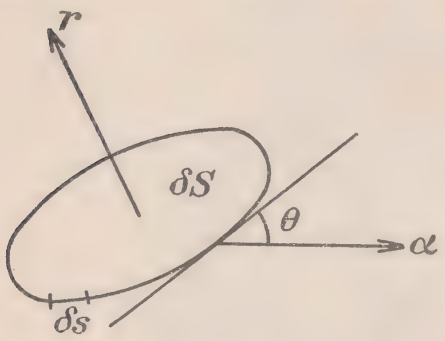


FIG. 22.

In some vector fields, however, the line integral round any closed path has a continuous variation with variation of the path. The vector field is then said to have a *curl*; it does not possess a potential.

Curl is a vector and may be resolved into component curls. If at a particular point in the field of a vector α the line integral $\oint \alpha \cos\theta \, ds$ be taken round a very small area δS (Fig. 22), then the component curl of α in the direction r, perpendicular to δS, is defined as

$$(\text{Curl } \alpha)_r = \underset{\delta S \to 0}{\text{Limit}} \frac{\oint \alpha \cos\theta \, ds}{\delta S} \quad . \quad . \quad . \quad (84)$$

and is a vector in the direction r. The conventional directions are related as those of a right-handed screw. In one direction the value is a maximum and this is the curl of the vector.

Equation 84 may be written as

$$(\text{Curl } \alpha)_r \, \delta S = \oint \alpha \cos\theta \, ds \quad . \quad . \quad . \quad . \quad (85)$$

If an extensive area (Fig. 23) be divided into small elements the sum of the line integrals for all elements is equal to the line integral round the boundary of an area since, in taking the line integrals round the separate elements each portion of the path which does not form part of the boundary of the total area is reckoned twice—in opposite directions. The sum of all elements represented on the left-hand side of equation 85 is the surface integral of the curl over the whole area. Thus the important result is obtained that the line integral of a vector round the boundary of a surface is equal to the surface integral of its curl over the whole area.

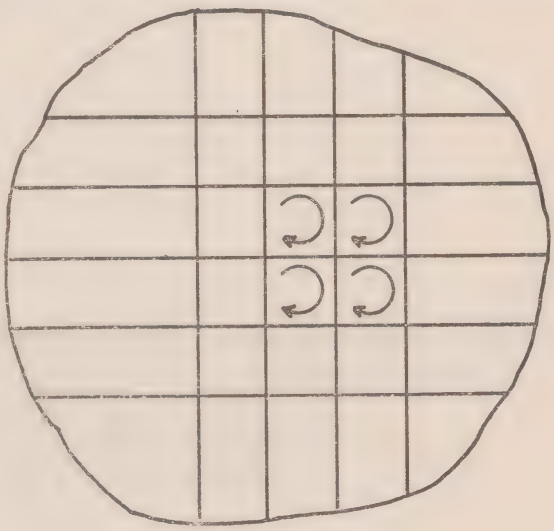


Fig. 23.

A convenient illustration is afforded by the variation of magnetic flux through an area bounded by a copper wire (e.g. a turn of wire round the limb of a transformer). If the area be in the plane of the paper (more or less—it need not be a plane area), a current circulating clockwise would produce a magnetic flux ϕ directed *into* the paper. A growth of flux into the paper therefore generates an e.m.f. which is anti-clockwise. The magnitude of this e.m.f. is $\frac{d\phi}{dt}$ and therefore

$$E = -\frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

If B is the flux density at any point in the area, ϕ is the surface integral of B over the area, and $\frac{d\phi}{dt}$ is the surface integral of $\frac{dB}{dt}$ throughout the area. The e.m.f. is the line integral of the electric force round the boundary of the area—the wire. Equation 86 is the result of the general relation (treated more fully later)

$$\text{Curl } \varepsilon = -\frac{dB}{dt} \quad . \quad . \quad . \quad . \quad . \quad (87)$$

The component curls of a general vector α along the three

co-ordinate axes can be readily determined. Taking right-handed axes * (Fig. 24), a small parallelepiped is imagined at the point considered. If the components of α are X, Y and Z,

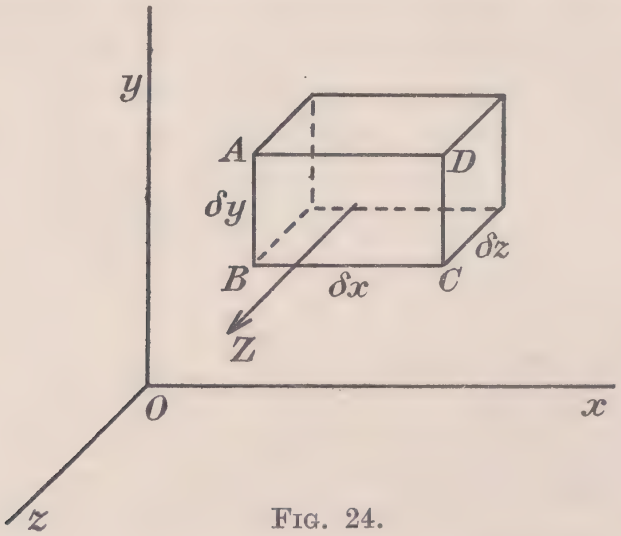


FIG. 24.

then the line integral round the face $ABCD$ is made up of the following parts :

$$\text{along AB, } -Y\,\delta y$$
$$\text{along BC, } X\,\delta x$$
$$\text{along } CD, \left(Y + \frac{\partial Y}{\partial x}\delta x\right)\delta y$$
$$\text{along } DA, -\left(X + \frac{\partial X}{\partial y}\delta y\right)\delta x$$

giving a total $$\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\delta x\,\delta y$$

The area $ABCD$ is $\delta x\ \delta y$. Hence

$$(\text{Curl } \alpha)_z = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

The other component curls are obtained similarly, giving

$$\left.\begin{aligned}(\text{Curl } \alpha)_x &= \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\\ (\text{Curl } \alpha)_y &= \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\\ (\text{Curl } \alpha)_z &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\end{aligned}\right\} \quad \ldots\ldots \quad (88)$$

* A right-handed system of axes is such that in turning from x to y a right-handed screw would travel in the z direction.

Magnetisation of Iron.

If a simple toroid consists of T turns uniformly wound upon a non-magnetic (say wooden) core, the radius of the ring being r cm. and the cross-sectional area A sq. cm. (Fig. 25), and if

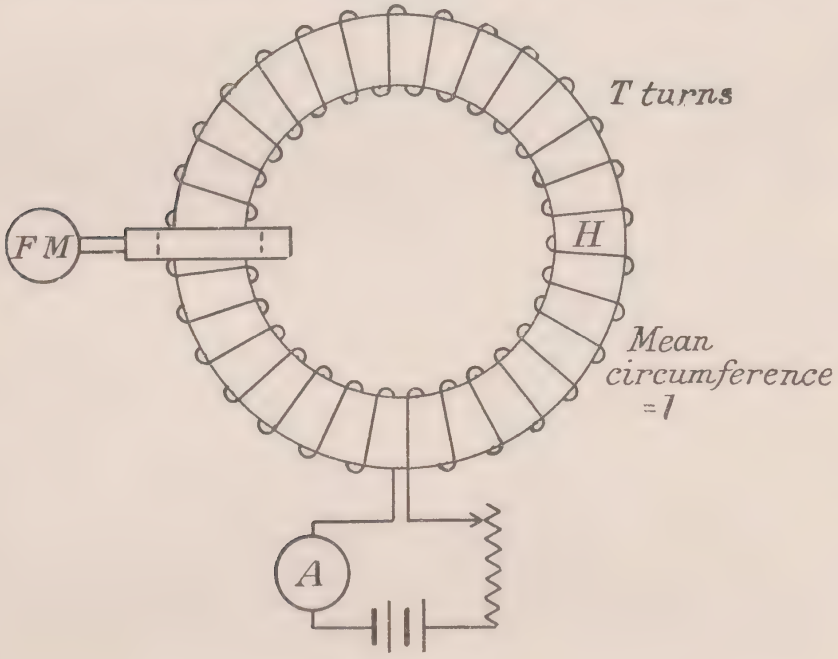


Fig. 25.

a current of I_a C.G.S. units be passed through the circuit, then the magnetomotive force along any path round the core is $4\pi \times$ total current linked $= 4\pi I_a T$. But this is equal to the line integral of the magnetic force round the ring, or Hl, where $l = 2\pi r$ (it is assumed that r is big compared with the dimensions of the cross-section), giving

$$H = \frac{4\pi I_a T}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

The total flux over the cross-section is $\phi = HA$. If a secondary coil, linked by this flux, is connected to a fluxmeter, simple reversal of the current shows that the flux is of this magnitude. If now a similar circuit be wound upon an iron ring and supplied with the same current the fluxmeter would indicate, on reversal of the current, that the flux had been enormously increased by the substitution of an iron core for a non-magnetic core. The flux density is increased μ times to some value B. μ is the *permeability* of the material (iron) and $B = \mu H$.

With some little difficulty, confirmatory results may be obtained with experiments upon rings having cavities cut in

particular ways. If a ring be prepared with a channel as indicated in Fig. 26, the force exerted upon a unit magnetic pole placed in the channel is H whether the ring be of wood or iron. If, however, the ring be slotted as in Fig. 27, a unit pole placed in the gap is acted upon by a force H if the ring is of wood. But if of iron the force is much greater than H but less than B as determined as above. As, however, the gap is made narrower and narrower the force increases towards the asymptotic value B as the gap becomes vanishingly narrow.*

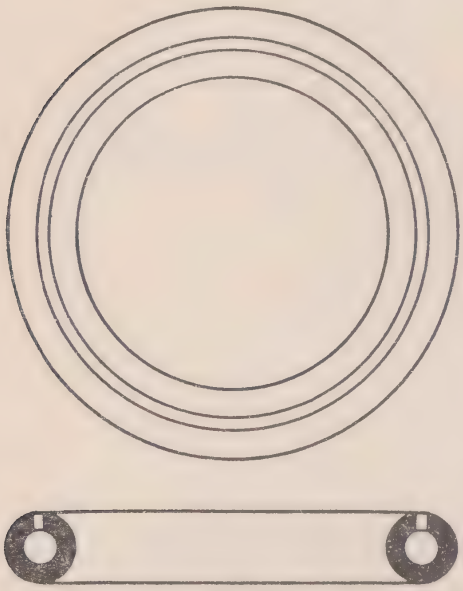


FIG. 26.

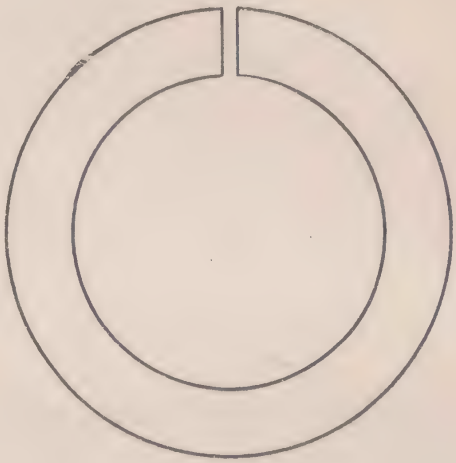


FIG. 27.

Ampere showed that this enormous effect of iron in increasing the flux could be explained if there were a circulation of electricity in the atoms of magnetic metals. Each atom would then behave as a small coil enclosing an area a in which a current I_a' is circulating.† Such a small coil tends to set itself so that the flux due to its own current is in the direction of the field in which it finds itself, and so the atoms tend to arrange

* This description is largely borrowed from Mallett and Vinycomb's *Foundations of Technical Electricity*.

† It is not assumed that this is the actual mechanism of the magnetic moment of the atom. Indeed the modern idea is that the magnetic moment is connected with electron spin rather than with the motion of the electron in an orbit. As the argument proceeds, it will be recognised that the increment in the line integral in penetrating an atom is dependent only on the magnetic moment of the atom and not upon the mechanism by which that moment is produced. The mechanism is undoubtedly electrical and it is natural to equate it to (note, not identify it with) a simple mechanism easily investigated.

themselves so that their flux lines form local groups or chains having no external effect. In a magnetic field these groups tend to be broken up and the atoms set themselves with their magnetic axes to some extent in the direction of the field. There is thus an increased flux as the atoms add their magnetomotive forces to the magnetomotive force of the original field.

Suppose now an elementary volume of the ring of Fig. 25 be considered having a length L in the direction of magnetisation and cross-sectional area A (Fig. 28), the number of magnetising turns associated with this elementary length being T. If such a path as $PQRS$ be considered the current linked is

$$I_a T + I_a' n$$

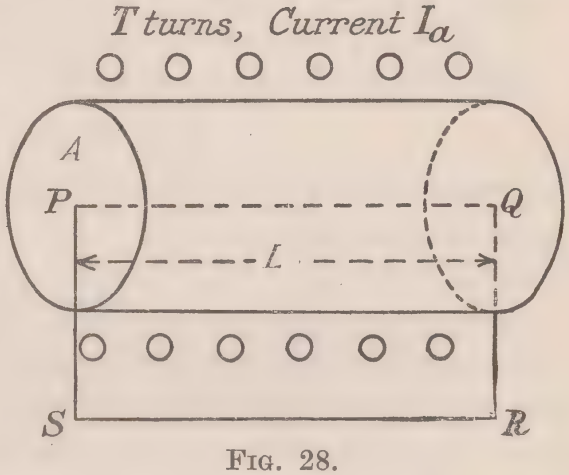


Fig. 28.

where n = number of atomic circuits pierced by PQ. n is readily determined as follows. The chance of any particular atomic circuit being pierced is a'/A, where a' is the projected area of the atomic circuit viewed in the direction of magnetisation. But the resolved part of the magnetic moment of the particular atomic circuit is $I_a'a' = m$, say (equation 69, p. 56). Hence the chance of a particular circuit being pierced is

$$\frac{a'}{A} = \frac{m}{I_a' A}$$

Since the total number of atomic circuits is very large it follows that n is equal to the sum of the resolved magnetic moments divided by $I_a'A$ or

$$n = M/I_a'A$$

where M is the magnetic moment of the elementary volume depicted. But $M = JAL$ (p. 52)

Hence $$n = \frac{JL}{I_a'} \text{ and } nI_a' = JL$$

Hence the total current linked by the path $PQRS$

$$= I_a T + JL \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

and 4π (total current linked) $= 4\pi I_a T + 4\pi JL$. (91)

Since in this elementary path the direction of the field is wholly along PQ, its strength being zero along QR, RS and SP

$$\oint \alpha \cos\theta \, ds = \alpha L = 4\pi I_a T + 4\pi J L. \quad . \quad . \quad (92)$$

or

$$\alpha = \frac{4\pi I_a T}{L} + 4\pi J \; . \quad . \quad . \quad . \quad . \quad (93)$$

α is B the total flux density,

$\frac{4\pi I_a T}{L}$ is H or that part of B due to the external magnetising current and is termed the magnetising force,

J is the intensity of magnetisation.

Equation 93 then becomes

$$B = H + 4\pi J \; . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

In equation 90 $I_a T$ are the external current turns initiating the magnetisation (the match that starts the fire), JL are the internal current turns, enormously greater than $I_a T$.

Ampere's assumption that the magnetisation of iron is fundamentally electrical in origin necessitates that lines of magnetic flux form closed curves, that the divergence of B is zero. It naturally follows that there is associated with any north pole an equal south pole. This is known to be an experimental fact.

A very helpful idea is expressed by Pidduck.* He considers H, the magnetic force, as that part of B which is derived from a potential. If a closed path within the iron be imagined and if at any point on the path the resolved value of B along the path be B_s and the resolved intensity of magnetisation be J_s, then the line integral of B round the path is $\oint B_s \, ds$. The value of this line integral depends upon the path, upon the number of atomic circuits linked. (The path is supposed not to link any external electrical circuit.) But the line integral expressed by $\oint (B_s - 4\pi J_s) ds$ is zero, and $H = B - 4\pi J$ (equation 94) is derived from a potential.

Units ; Laws of Electrodynamics.

Two completely different systems of units have so far been employed—the C.G.S. electrostatic units and the C.G.S. electromagnetic units.

* *Mathematical Theory of Electricity*, pp. 62 and 63.

The C.G.S. electrostatic units are based upon the definition of unit electric charge. The C.G.S. electromagnetic units are based upon the definition of unit magnetic pole. From this unit magnetic pole is derived the electromagnetic unit of current and hence the electromagnetic unit of quantity of electricity, or charge. The electrostatic and electromagnetic units of charge are of enormously different magnitude. It has been shown (p. 57) that if q electromagnetic units of charge move with a velocity of u cm. per second the intensity of the magnetic field produced at a distance of 1 cm. perpendicular to the path is qu. One electromagnetic unit of charge (10 coulombs) moving with a velocity of 1 cm. per second produces unit field at a distance of 1 cm. Unit electrostatic charge would have to move with a velocity of c cm. per second to produce unit magnetic field at a distance of 1 cm. This velocity c is found to be 3×10^{10} cm. per second—the velocity of light. At a distance of 1 cm. from a charge of q electrostatic units moving with a velocity u the magnetic field is therefore

$$H = \frac{qu}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

But at 1 cm. from a charge q the electric displacement $D = \frac{q}{4\pi}$.

Hence

$$H = 4\pi D \cdot \frac{u}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

A closed area may be considered (Fig. 29) through which the electric flux N is increasing. The density at a point P is D and Faraday tubes are crossing the boundary with a component velocity u along n. The rate at which tubes are entering along a length δs is $Du\,\delta s$.

Hence

$$\frac{dN}{dt} = \oint Du\,ds = \frac{c}{4\pi}\oint H\,ds \text{ (from equation 96)}$$

$$= \frac{c}{4\pi} \times \text{(m.m.f. round the path)}$$

or,

$$\text{the m.m.f. round the path} = \frac{4\pi}{c} \cdot \frac{dN}{dt} \quad . \quad . \quad . \quad . \quad . \quad (97)$$

But this is the m.m.f. produced by a current equal to $\frac{1}{c}\frac{dN}{dt}$. This is *Maxwell's displacement current.* Such a current behaves magnetically in exactly the same way as a conduction current.

It has already been seen (p. 58) that when magnetic flux cuts a conductor an e.m.f. per unit of length is generated equal to the rate of cutting by magnetic tubes. No material conductor need actually be considered. If magnetic flux is moving, an electric force is produced at right angles to the flux and to its direction of motion. The magnitude of this electric force is Bu (in electromagnetic units). But the unit of electric force in the electrostatic system is c times that in the electromagnetic system (since the same work, 1 erg, is done when unit charge is moved through unit difference of potential in either system). Hence if ε be measured in electrostatic units

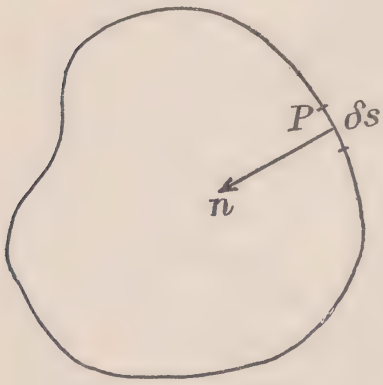


FIG. 29.

$$\varepsilon = \frac{Bu}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

and also the

$$\text{e.m.f. round any closed curve} = -\frac{1}{c}\frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad (99)$$

(in electrostatic units)

Numerical Relations between the Units.

The practical units are based upon the C.G.S. electromagnetic units. The ampere is $\frac{1}{10}$th of the C.G.S.E.M.U. of current and the coulomb $\frac{1}{10}$th of the C.G.S.E.M.U. of quantity and 3×10^9 times the unit electrostatic charge. The volt is so chosen that 1 volt $\times$ 1 coulomb = 1 joule = 10^7 ergs. Hence it follows that a volt is 10^8 E.M.U. of potential. One volt is generated when magnetic lines are cut at the rate of 10^8 per second. The E.S.U. of potential is 3×10^{10} times the E.M.U. of potential and is therefore equal to 300 volts. The practical unit of inductance, the henry, is such that 1 volt causes a rate of change of current of 1 ampere per second. It is therefore 10^9 times the E.M.U. of inductance (the cm.). Similarly the ohm is 10^9 times the E.M.U. unit of resistance.

The E.M.U. of capacitance is such that 10 coulombs is stored with a difference of potential of 10^{-8} volts; it is therefore equal to 10^{9} farads! The E.S.U. of capacitance (the cm.) is such that $\frac{1}{3 \times 10^{9}}$ coulombs is stored under a difference of potential of 300 volts and it is therefore equal to $\frac{1}{9 \times 10^{11}}$ farads (roughly a micro-micro-farad). When using practical units the necessary ciphers must be inserted in the formulæ based upon the electromagnetic units. In many cases it is convenient to work in C.G.S. electromagnetic units.

CHAPTER VI

ELECTRICAL PROBLEMS

In this chapter it is proposed to deal with a number of problems involving the mathematical, and some of the electromagnetic, principles treated in Chapters I–V.

Average Value of a Sinoidal Function.

By the average value of a function $y = f(x)$ over a range of values of x from, say, x_1 to x_2 is meant

$$\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)dx$$

In Fig. 30, $f(x)$ is plotted against x. The area of strip indicated is $y\ \delta x$. The area of the figure between the abscissæ x_1 and x_2 is $\int_{x_1}^{x_2} y\ dx$. The average height of the figure is therefore $\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} y\ dx$. In the case of a sine function the average that is required is the numerical average between two successive nodes. If the function be expressed as $y = A \sin \omega t$; it is required to determine the average between $\omega t = 0$ and $\omega t = \pi$, or between $t = 0$ and $t = \pi/\omega$. This average is

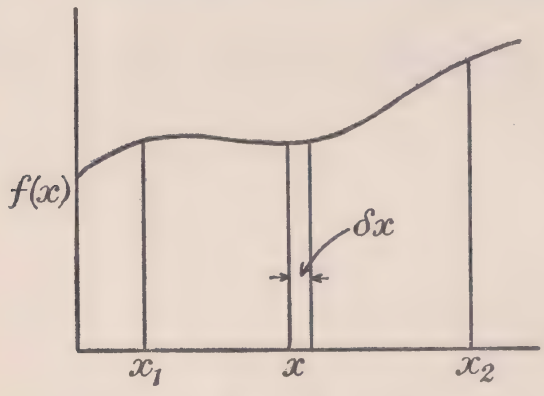


FIG. 30.

$$\frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} A \sin \omega t\ dt$$

$$= \frac{\omega}{\pi}\left[-\frac{A}{\omega}\cos \omega t \right]_0^{\frac{\pi}{\omega}} = \frac{2}{\pi}A$$

The average value is therefore $\frac{2}{\pi}$ times the maximum value.

Root Mean Square Value of a Sinoidal Function.

By the root mean square (R.M.S.) value of a function is meant the square root of the average value of the square of the function. It is clearly this particular mean value which determines the average rate of heating produced by an electric current. In the case of a sine function the R.M.S. value is obviously the same for the positive and negative halves of the wave and the value determined for the positive half of the wave is applicable to any number of complete waves. If the square of the sine function be expressed as $y = A^2 \sin^2 \omega t$, the average value of y over a half wave is

$$\frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} A^2 \sin^2 \omega t \, dt = \frac{\omega A^2}{2\pi}\int_0^{\frac{\pi}{\omega}} (1 - \cos 2\omega t) dt$$

$$= \frac{\omega A^2}{2\pi}\Big[t - \frac{1}{2\omega}\sin 2\omega t\Big]_0^{\frac{\pi}{\omega}} = \frac{A^2}{2}$$

The R.M.S. value is therefore $\frac{A}{\sqrt{2}}$, or $\frac{1}{\sqrt{2}}$ times the maximum value.

Intensity of Magnetic Field near to a Straight Wire carrying a Current.

Two methods of determining the intensity of the magnetic field H at a distance r from a long straight wire conveying a current I_a C.G.S. units are of interest. The simpler method is to consider the wire as part of a very large circuit. All but the long straight part under consideration is supposed to be so far away that the magnetic field due to it is negligible. By symmetry the lines of magnetic force are circles concentric with the wire. The line integral round any line of force, of radius r, is therefore

$$\oint H \,.\, ds = H \,.\, 2\pi r$$

But by equation 80 (p. 62) the line integral $= I_a \,.\, 4\pi$.

Hence $$H = \frac{2I_a}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

If the current is I amperes then

$$H = \frac{2I}{10r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

The same result can be obtained with slightly greater difficulty but with more certain validity by use of equation 67 (p. 54). It will be worked out in this way since the method is of wide application. If at a point P (Fig. 31) a unit magnetic pole be placed the elementary force δF exerted by the circuit element δl upon the pole is

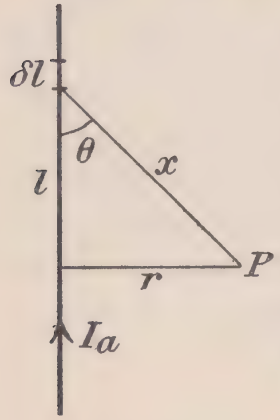


FIG. 31.

$$\delta F = I_a \frac{\delta l}{x^2} \sin\theta$$

but $x = r \operatorname{cosec}\theta$ and $l = r\cot\theta$

giving $\delta l = -r \operatorname{cosec}^2\theta\,\delta\theta$

whence $$\delta F = -I_a \frac{\sin\theta}{r}\delta\theta$$

Hence the force exerted upon the pole is

$$F = H = \int_{\pi}^{0} -I_a \frac{\sin\theta}{r} d\theta = \frac{2I_a}{r}$$

Intensity of the Magnetic Field within a Solid Cylindrical Conductor.

Within a solid cylindrical conductor the value of H is less than at the surface. The lines of force are still circles. At a radius r, less than R the radius of the wire, the line integral

$$\oint H \,.\, ds = H.2\pi r$$

The current enclosed within the line of force is $I_a \frac{r^2}{R^2}$ where I_a is the total steady current in the wire. Hence by equation 80 (p. 62)

$$\oint H \,.\, ds = 4\pi I_a \frac{r^2}{R^2} = 2\pi r H$$

or $$H = 2I_a \frac{r}{R^2}$$

Magnetic Field due to a Circular Current.

A current I_a flows in a conductor bent into a circle of radius r. It is required to determine the intensity of the magnetic field at a point on the axis distant x from the centre (Fig. 32). By symmetry

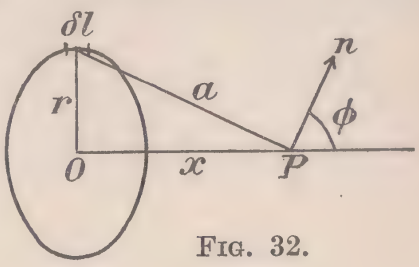


FIG. 32.

the magnetic force must be directed along the line OP. The intensity of this force can be determined by the use of equation 67 (p. 54), as for a straight conductor. Due to an element δl of the circuit the elementary force exerted at P is $\delta F = I_a \frac{\delta l}{a^2}$, since a is normal to the element δl. This force is directed normally to a in the direction n. The component force in the direction OP is

$$\delta F' = \delta F \cos \phi = \delta F \frac{r}{a} = I_a \frac{r}{a^3} \delta l$$

The total force

$$F' = H = I_a \frac{r}{a^3} \oint dl = \frac{2\pi r^2}{a^3} I_a \quad . \quad . \quad . \quad (102)$$

Another method of obtaining the same result involves the use of the magnetic potential $V = I_a \Omega$ (equation 70, p. 57). To apply this it is first necessary to determine the solid angle subtended at P by the circuit. This may be determined simply from geometrical considerations. If with centre P a sphere of radius a be described (Fig. 33) the area of the spherical cap is $2\pi ah = 2\pi a\ (a - x)$. This spherical cap subtends the same solid angle at P as the circuit. Hence

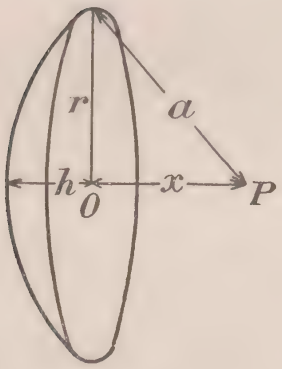


Fig. 33.

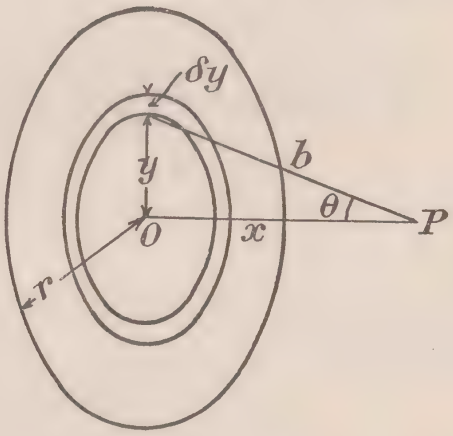


Fig. 34.

$$\Omega = \frac{2\pi ah}{a^2} = 2\pi\left(1 - \frac{x}{a}\right)$$

As it is not generally possible to find a simple geometrical method of determining the solid angle subtended by a circuit at a given point, Ω will also be determined by integration. Considering an element of the plane of the circuit bounded between two circles of radius y and $y + \delta y$ (Fig. 34), the solid

angle $\delta\Omega$ subtended at P is

$$\delta\Omega = \frac{2\pi y\,\delta y\cos\theta}{b^2}$$

but $b^2 = x^2 + y^2$ and $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ giving

$$\delta\Omega = \frac{2\pi\, yx}{(x^2+y^2)^{\frac{3}{2}}}\delta y$$

Hence $\Omega = \displaystyle\int_0^r \frac{2\pi\, yx}{(x^2+y^2)^{\frac{3}{2}}}dy = 2\pi x\Big[_0^r - \frac{1}{\sqrt{x^2+y^2}}\Big]$

$$= 2\pi x\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+r^2}}\right) = 2\pi\left(1 - \frac{x}{a}\right)$$

The intensity of the magnetic force is immediately determined

$$H = -\frac{dV}{dx}$$

$$V = 2\pi I_a\left(1 - \frac{x}{a}\right) = 2\pi I_a\left(1 - \frac{x}{\sqrt{x^2+r^2}}\right)$$

whence $H = \dfrac{2\pi I_a r^2}{(x^2+r^2)^{\frac{3}{2}}} = \dfrac{2\pi r^2}{a^3}I_a$

Magnetic Force due to a Circular Current at a Point in its Plane.

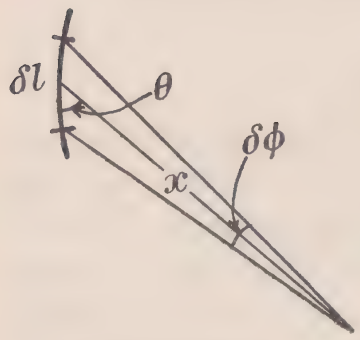


FIG. 35.

At a point in the plane of a circuit the relation expressed by equation 67 (p. 54) may be simplified. If x be the distance of the point from the circuit element then (Fig. 35)

$$\delta l \sin\theta = x\,\delta\phi$$

and hence

$$\delta F = \frac{I_a}{x}\delta\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

and for the closed circuit

$$H = I_a\oint\frac{d\phi}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

At the centre of the circle

$$H = I_a\int_0^{2\pi}\frac{d\phi}{r} = \frac{2\pi I_a}{r}$$

To determine the intensity of the magnetic field at a point away from the centre is more difficult. In Fig. 36

$$x = \sqrt{r^2 - a^2 \sin^2 \phi} + a \cos \phi$$

$$\frac{1}{x} = \frac{\sqrt{r^2 - a^2 \sin^2 \phi} - a \cos \phi}{r^2 - a^2}$$

whence

$$H = \frac{I_a}{r^2 - a^2}\left[\int_0^{2\pi} \sqrt{r^2 - a^2 \sin^2 \phi}\, d\phi - \int_0^{2\pi} a \cos \phi\, d\phi\right]$$

The value of the second integral is obviously zero and hence

$$H = \frac{I_a}{r^2 - a^2}\int_0^{2\pi} \sqrt{r^2 - a^2 \sin^2 \phi}\, d\phi \quad . \quad . \quad (105)$$

An examination of this integral shows that the values of $\sin^2 \phi$ are repeated in all four quadrants and hence we may write

$$H = \frac{4I_a}{r^2 - a^2}\int_0^{\frac{\pi}{2}} \sqrt{r^2 - a^2 \sin^2 \phi}\, d\phi$$

or

$$H = \frac{4I_a}{r(1 - k^2)}\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi \quad . \quad . \quad (106)$$

where $k = a/r$.

This integral cannot be determined by any of the standard methods so far given. If the expression $\sqrt{1 - k^2 \sin^2 \phi}$ be expanded by the binomial theorem, a convergent series is obtained which can be integrated term by term. Sufficient terms are included to give the result to the accuracy required. It is one of the *elliptic integrals* and is usually denoted by E. It occurs fairly frequently and it has been tabulated for various values of the argument k. When an expression of the form given in equation 105 is obtained it must be thrown into the form of equation 106. Then

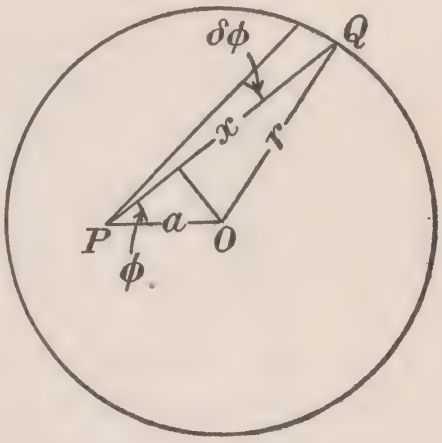


FIG. 36.

$$E(k) \equiv \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi \quad . \quad . \quad . \quad (107)$$

which can be looked up in tables for the particular value of k.

To take a particular case $E(\frac{1}{2}) = 1{\cdot}4675$. The value of H half-way between the centre and circumference of a circle is

$$H = \frac{4I_a}{r(1 - 0{\cdot}25)} \times 1{\cdot}4675 = \frac{7{\cdot}83I_a}{r}$$

This is about 25 per cent. greater than the value at the centre of the circle ($6{\cdot}28I_a/r$).

Intensity of the Field at the Centre of a Square.

The intensity of the field at any point within a rectangle is readily determined by the method used for the long straight wire (p. 76). Each side is treated separately. The limits of integration are between θ_1 and θ_2 (Fig. 37) instead of between 0 and π. Thus at the centre of a square $\theta_1 = 45°$ and $\theta_2 = 135°$. If p is the perpendicular distance from the centre of the square to one side, the intensity of the field due to one side is $\frac{I_a\sqrt{2}}{p}$ and therefore

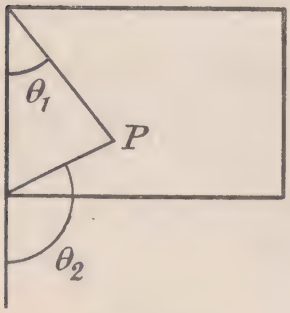


Fig. 37.

$$H = \frac{4\sqrt{2}I_a}{p} \quad . \quad . \quad . \quad . \quad . \quad (108)$$

Forces between Conductors conveying Currents.

Any difficulties which arise in determining the forces between conductors are purely analytical ones. The general method is quite simple and involves first finding the magnetic field produced by the first conductor at any point on the second. The force on an element of the second conductor can then be determined, and the total force by integration throughout the length of the second conductor. Certain simple cases are of interest and practical importance.

Force between two Indefinitely Long Parallel Conductors.

At a distance r from a long straight conductor carrying a current I_a C.G.S. units, the intensity of the magnetic field is $H = 2I_a/r$. On a centimetre length of a parallel conductor

at this distance, carrying a current I_a', the force is

$$F = \frac{2I_aI_a'}{r} \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad (109)$$

If the currents are in the same direction the force is one of attraction.

Pinch Effect.

Considering a conductor as made up of a large number of parallel filaments it is seen that the presence of a current in the conductor produces a mechanical pressure tending to reduce the cross-section. This effect increases with the current density. In a liquid conductor this may produce a condition of instability. A casual reduction of cross-section at a particular place may, by increasing the current density, bring about a further reduction of cross-section in spite of the restoring forces, and the conductor may actually be ruptured by this *pinch effect*. The following example is interesting : " If a cylindrical column of mercury 1 cm. diameter carries a current of 100 amperes, calculate the intensity of the mechanical pressure due to pinch effect (i) at a radius of 0·25 cm. ; (ii) at the axis of the conductor. Find also the total axial mechanical force arising from this effect " (L.U. Ext. E.M. and M.I., I. 2, 1934, part question).

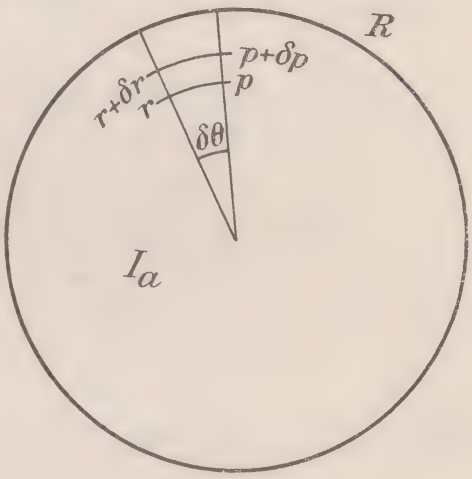


FIG. 38.

This example will be worked out in general terms. The column will be supposed to be carrying a current I_a absolute units. The equilibrium of the element enclosed between $\delta\theta$ and r and $r + \delta r$ (Fig. 38) will be considered. At the radius r,

$$H = \frac{2I_a}{r} \cdot \frac{r^2}{R^2} = \frac{2I_ar}{R^2}$$

The current carried by the element is $\frac{I_ar\ \delta\theta\ \delta r}{\pi R^2}$. Hence the elementary force urging 1 cm. length of this filament towards the axis is

$$\delta F = \frac{2I_a{}^2r^2\ \delta\theta\ \delta r}{\pi R^4}$$

This force acts upon an area $r\,\delta\theta$, hence

$$-\delta p = \frac{\delta F}{r\,\delta\theta} = \frac{2I_a{}^2 r\,\delta r}{\pi R^4}$$

where p is the pressure at radius r.
Hence

$$p = \int_r^R \frac{2I_a{}^2 r}{\pi R^4} dr = \frac{I_a{}^2}{\pi R^2}\left(1 - \frac{r^2}{R^2}\right) \quad . \quad . \quad . \quad (110)$$

At any point the pressure is the same in all directions. The axial force upon the annulus r, $r + \delta r$ is therefore

$$2\pi r p\,\delta r$$

The total force on the cross-section is therefore

$$P = \int_0^R 2\pi r p\,dr = \frac{I_a{}^2}{\pi R^2}\int_0^R 2\pi r\left(1 - \frac{r^2}{R^2}\right)dr = \frac{I_a{}^2}{2} \text{ dynes} \quad (111)$$

In the particular example $I_a = 10$, hence

(i) p at 0·25 cm. is

$$p_1 = \frac{100}{0{\cdot}7854}(1 - \tfrac{1}{4}) = 95{\cdot}5 \text{ dynes per sq. cm.}$$

and (ii) p on the axis is

$$p_2 = \frac{100}{0{\cdot}7854} = 127{\cdot}3 \text{ dynes per sq. cm.}$$

The total axial force is 50 dynes.

Calculation of Self-inductance of a Circuit.

The self-inductance of a circuit in centimetres is the number of flux linkages for unit C.G.S. current (10 amperes). In henries the self-inductance is the number of flux linkages per ampere divided by 10^8. In some cases there is little difficulty in calculating the self-inductance of a circuit using the principles already given. In obtaining accurate formulæ it is necessary to take into account the flux linkages within the wire. If the wire is of radius R and carries a current I_a the current enclosed within a radius r is $I_a r^2/R^2$ and the value of H at a radius r is $2I_a r/R^2$. Per unit length of conductor the element of flux between r and $r + \delta r$ enclosing the current $I_a\frac{r^2}{R^2}$ is $\frac{2I_a r}{R^2}\delta r$.

The total current linkages are therefore $\int_0^R \frac{2I_a{}^2 r^3}{R^4} dr = \frac{1}{2}I_a{}^2$.

But the current linkages are $LI_a{}^2$. Hence $L = \frac{1}{2}$ cm. per cm.

If the conductor has a length l and a permeability μ then the inductance in centimetres is

$$L = \tfrac{1}{2}\mu l \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

In many cases it is unnecessary to take this contribution to the inductance into account.

Inductance of a Coaxial Cable.

The radius of the inner conductor of a coaxial cable is assumed to be r_1 and of the outer conductor r_2. If the current carried by the inner conductor is I_a (and by the outer conductor $-I_a$) there is a magnetic field only within the outer conductor. At a radius $r > r_1$ and $< r_2$ the intensity of the magnetic field is $2I_a/r$. The flux enclosed between radii r and $r + \delta r$ per unit length of the conductor is $\delta\phi = \dfrac{2I_a}{r}\delta r$. If the field within the inner conductor be neglected the total flux linking this conductor per unit length is

$$\phi = \int_{r_1}^{r_2} \frac{2I_a}{r} dr = 2I_a \log_e \frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad (113)$$

Hence the inductance per unit length is

$$L = 2 \log_e \frac{r_2}{r_1} \text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

or in henries

$$L = 2 \times 10^{-9} \log_e \frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad (115)$$

Inductance of a Pair of Parallel Wires.

Considering the loop formed by unit length of the two conductors indicated in Fig. 39, the flux due to the current in conductor A threading the loop is, by equation 113, $2I_a \log_e \dfrac{d}{r}$ (the radius of B is neglected in comparison with d and the internal flux in the conductor A is also neglected). Since the current in conductor B gives an equal flux linking the loop the inductance per unit length is

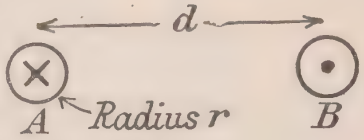


FIG. 39.

$$L = 4 \log_e \frac{d}{r} \text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

or

$$L = 4 \times 10^{-9} \log_e \frac{d}{r} \text{ henries} \quad . \quad . \quad . \quad (117)$$

Capacitance of a Coaxial Cable.

The lines of electric force stretching from the inner to the outer conductor of a coaxial cable are radial and in planes perpendicular to the axis. It therefore follows that if the charge per unit length of the inner conductor is q the density of Faraday tubes, or the displacement, at a radius r is

$$D = \frac{q}{2\pi r}$$

Hence the electric force is

$$\varepsilon = \frac{4\pi D}{\kappa} = \frac{2q}{\kappa r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

The difference of potential between the two conductors is

$$V = \int_{r_1}^{r_2} \frac{2q}{\kappa r} dr = \frac{2q}{\kappa} \log_e \frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad (119)$$

and the capacitance in centimetres is therefore

$$C = \frac{\kappa}{2 \log_e \frac{r_2}{r_1}} \quad . \quad . \quad . \quad . \quad . \quad (120)$$

In farads the capacitance is

$$C = \frac{1}{9 \times 10^{11}} \cdot \frac{\kappa}{2 \log_e \frac{r_2}{r_1}} \quad . \quad . \quad . \quad . \quad (121)$$

Capacitance of a Pair of Parallel Wires (Approximate Solution).

Two line charges of $+q$ and $-q$ per unit length, the lines being separated by a distance d, are considered. In the immediate neighbourhood of one line charge, where the influence of the other is relatively but feebly felt, the equipotential surfaces are circular cylinders having the line charge as axis.

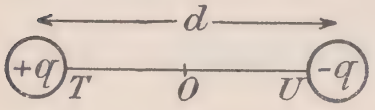


FIG. 40.

If an equipotential surface of radius r be considered round each line charge, r being much less than d, the case of two conductors of radius r and separated by a distance d is obtained (Fig. 40). Due to the charge $+q$ the difference of potential between 0 (the mid-point between the conductors) and T is, from equation 119, $\frac{2q}{\kappa} \log_e \frac{d/2}{r}$, and due to the charge $-q$ the

difference of potential is $\frac{2q}{\kappa}\log_e\frac{d-r}{d/2}$. Hence, since by symmetry the potential of 0 must be zero, the potential of the positively charged line is $\frac{2q}{\kappa}\log_e\frac{d-r}{r}$ which, if $r << d$, may be written as $\frac{2q}{\kappa}\log_e\frac{d}{r}$. The difference of potential between the two conductors is

$$V = \frac{4q}{\kappa}\log_e\frac{d}{r} \quad . \quad . \quad . \quad . \quad . \quad (122)$$

and the capacitance in centimetres is

$$C = \frac{\kappa}{4\log_e\frac{d}{r}} \quad . \quad . \quad . \quad . \quad . \quad (123)$$

In farads the capacitance is

$$C = \frac{1}{9\times10^{11}}\cdot\frac{\kappa}{4\log_e\frac{d}{r}} \quad . \quad . \quad . \quad . \quad (124)$$

This approximation is usually sufficient. When, however, the separation is small the solution given in the next paragraph must be used.

Capacitance of a Pair of Parallel Wires (Exact Solution).

The line charges $+q$ and $-q$ are supposed separated by a distance l (Fig. 41). The potential at O is zero. At a point S the potential is

$$\frac{2q}{\kappa}\left(\log_e\frac{OP}{PS}+\log_e\frac{QS}{OP}\right)=\frac{2q}{\kappa}\log_e\frac{r_2}{r_1}$$

If S is on an equipotential surface then

$$\frac{2q}{\kappa}\log_e\frac{r_2}{r_1}=\text{constant}$$

or

$$\frac{r_2}{r_1}=\text{constant} \quad . \quad . \quad . \quad . \quad (125)$$

Equation 125 is the equation to a circular cylinder. For the equipotential surface so defined a conductor may be substituted, and a similar conductor, similarly disposed with

respect to Q, forms the pair. It follows that the potential of the left-hand conductor is

$$\frac{2q}{\kappa}\log_e\frac{QT}{PT}=\frac{2q}{\kappa}\log_e\frac{l-a}{a}$$

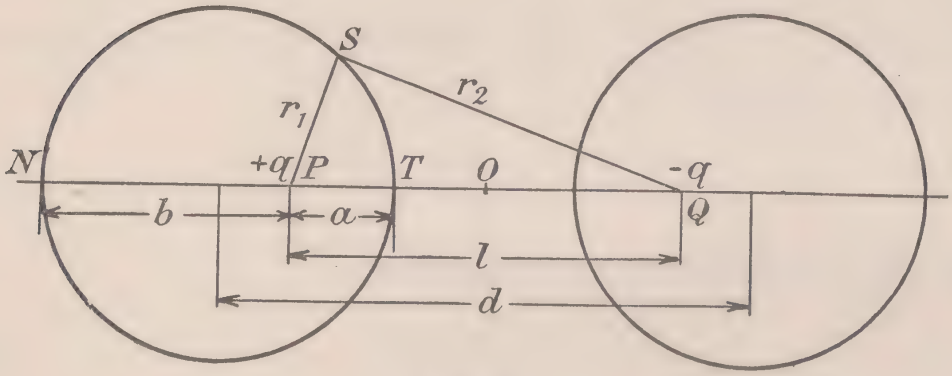


FIG. 41.

The difference of potential between the two conductors is

$$V=\frac{4q}{\kappa}\log_e\frac{l-a}{a} \quad . \quad . \quad . \quad . \quad . \quad (126)$$

and the capacitance in centimetres is

$$C=\frac{\kappa}{4\log_e\dfrac{l-a}{a}} \quad . \quad . \quad . \quad . \quad . \quad (127)$$

It remains to express l and a in terms of d and r. Taking the collinear values of r_1 and r_2 the following relation is obtained

$$\frac{QT}{PT}=\frac{QN}{PN} \text{ or } \frac{l-a}{a}=\frac{l+b}{b}$$

Since $a+b=2r$, $\quad \dfrac{d+a-2r}{a}=\dfrac{d+a}{2r-a}$

giving $$a=\frac{\sqrt{d^2-4r^2}-(d-2r)}{2}$$

and $$l=\sqrt{d^2-4r^2}$$

whence the capacitance in centimetres is

$$C=\frac{\kappa}{4\log_e\dfrac{\sqrt{d^2-4r^2}+(d-2r)}{\sqrt{d^2-4r^2}-(d-2r)}} \quad . \quad . \quad (128)$$

This reduces to (123) when $r << d$.

Dielectric Stress in a Coaxial Cable ; Optimum Size of Central Conductor.

From equations 118 and 119 relating to the coaxial cable the electric force, or potential gradient, or dielectric stress, at a distance $r(> r_1$ and $< r_2)$ from the axis is

$$\varepsilon = \frac{V}{r \log_e \frac{r_2}{r_1}}$$

This stress is a minimum at the radius r_2 and a maximum at the radius r_1 where it becomes

$$\varepsilon_1 = \frac{V}{r_1 \log_e \frac{r_2}{r_1}} \quad . \quad . \quad . \quad . \quad . \quad (129)$$

If r_2 be fixed the maximum stress depends upon r_1. If r_1 be made nearly equal to r_2 the value of ε_1 is big because of the small distance between the two conductors. If r_1 is made very small the stress at the surface of r_1 is great because of the high surface density. The optimum value of r_1 is that which gives a minimum value for ε_1. This occurs when $r_1 \log_e \frac{r_2}{r_1}$ is a maximum.

If $x = r_1 \log_e \frac{r_2}{r_1} = r_1(\log_e r_2 - \log_e r_1)$

$$\frac{dx}{dr_1} = \log_e \frac{r_2}{r_1} - 1$$

Equating this to zero defines the optimum value

and gives $\log_e \frac{r_2}{r_1} = 1$ or

$$r_1 = \frac{r_2}{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

(e here, of course, is the exponential function.)

Optimum Dimensions for a High-frequency Coaxial Feeder.

The best dimensions for a high-frequency coaxial feeder are determined by the attenuation. It is shown later (p. 278) that the attenuation is defined by $R/2Z_0$ where R is the resistance per unit length and Z_0 the *characteristic impedance* (p. 276)

of the line. It is also shown (p. 278) that at high frequencies $Z_0 = \sqrt{L/C}$ where L is the inductance and C the capacitance per unit length. It is further shown (p. 249) that at high frequencies the conductance is proportional to the periphery of the conductor. For a given external radius r_2 the value of r_1 giving a minimum attenuation has to be determined. From equations 114 and 120

$$Z_0 = \sqrt{L/C} \propto \log_e \frac{r_2}{r_1}$$

The resistance is the sum of the resistances of the inner and outer conductors or

$$R \propto \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

The optimum value of r_1 is given when

$$x = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \Big/ \log_e \frac{r_2}{r_1} \text{ is a minimum}$$

If $y = r_2/r_1$

$$r_2 x = \frac{y+1}{\log_e y} \text{ and } \frac{d}{dy}(r_2 x) = \frac{\log_e y - \dfrac{y+1}{y}}{(\log_e y)^2}$$

Equating this to zero gives

$$\log_e y - \frac{y+1}{y} = 0 \quad . \quad . \quad . \quad . \quad (131)$$

This equation is best solved by the application of Taylor's theorem (p. 27). It is soon found that y is between 3 and 4. Taking 3·5 as an approximate solution and writing

$$f(y) = \log_e y - \frac{y+1}{y}$$

$$f'(y) = \frac{1}{y} + \frac{1}{y^2}$$

If $y = 3{\cdot}5$, $f(y) = -0{\cdot}033$ and $f'(y) = 0{\cdot}367$,

$$f(y)/f'(y) = 0{\cdot}09, \text{ whence } \frac{r_2}{r_1} = 3{\cdot}59.$$

Pull of a Solenoid upon an Iron Plunger. Moving Iron Ammeter.

This problem is a particular case of a much more general one involving the forces operating between electric circuits and

between electric circuits and masses of permeable material. The method of tackling the problem is to imagine a small mechanical displacement to be permitted and to determine the consequent energy relations. The displacement involves (*a*) mechanical work being done (positive or negative), (*b*) electrical energy being supplied, and (*c*) an increase in energy of the system. It is clear that

Electrical energy supplied = mechanical work done + increase in energy of the system . . . . . (132)

The particular case considered is shown in Fig. 42 (C.G.S. units will be used ; conversion to practical units is easy if required). A coil A, supposed resistanceless, is supplied with a continuous current I_a from the generator G (the e.m.f. developed by this generator is imagined to vary in such a way that the current I_a is maintained constant under all circumstances). The position of an iron plunger B is defined by the distance x from some fixed point C. The force P exerted upon the plunger acts in the direction of x increasing. In this position the coil A possesses an inductance L and the energy of the system is $\frac{1}{2}LI_a^2$. The plunger is permitted to move a small distance δx ; the inductance increases to $L + \delta L$ and the energy to $\frac{1}{2}(L + \delta L)I_a^2$. The increase of energy is $\frac{1}{2}I_a^2\,\delta L$. During the movement, increasing the flux through the coil, energy must be supplied by the generator equal to $I_a(I_a\,\delta L) = I_a^2\,\delta L$. The mechanical work done is $P\,\delta x$. Hence from equation 132

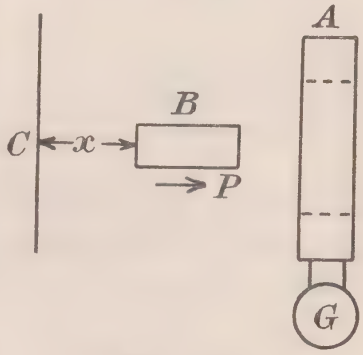


Fig. 42.

$$I_a^2\,\delta L = \tfrac{1}{2}I_a^2\,\delta L + P\,\delta x \quad . \quad . \quad . \quad . \quad (133)$$

or

$$P = \tfrac{1}{2}I_a^2\frac{dL}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

It must be remarked that this result is too commonly obtained by equating the change in energy of the system to the mechanical work done. This gives the correct result, but only because the system receives from the generator an amount of energy equal to double either of these quantities.

This attraction of a coil for a piece of iron is the principle of one type of ammeter, the attractive force being balanced

by a spring or by gravity. In this case the iron is pivoted upon an arm of length r. The torque T acting is then Pr; $\delta x = r\,\delta\theta$ where $\delta\theta$ is the small angular movement of the arm. Equation 134 then becomes

$$T = Pr = \tfrac{1}{2}I_a{}^2\left(r\frac{dL}{dx}\right)$$

or

$$T = \tfrac{1}{2}I_a{}^2\frac{dL}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad (135)$$

The following example illustrates the application of this.

"The relationship between the inductance of a moving iron ammeter, the current, and the position of the pointer is as follows:

Reading (amperes) . .	0·8	1·0	1·2	1·4	1·6	1·8	2·0
Deflection (degrees) . .	16·0	26·0	36·5	49·5	61·5	74·5	86·5
Inductance μH . . .	573·2	574·2	575·2	576·6	577·8	578·8	579·5

Calculate the deflecting torque when the current is (*a*) one ampere, (*b*) two amperes" (L.U. Ext. E.M. and M.I., I. 2, 1936, part question).

From the table given the values of $\frac{dL}{d\theta}$ may be obtained either by estimation or from a curve. An examination shows that the limitation of measurement is such that the quantities required cannot be obtained with great accuracy and it is best to use a curve of $\frac{dL}{d\theta}$ against the current. Thus between 1·4 and 1·6 amperes, $\Delta\theta$ is 12° and ΔL is 1·2 μH giving approximately for 1·5 amperes, $\frac{dL}{d\theta} = 0{\cdot}100$ μH/degree.

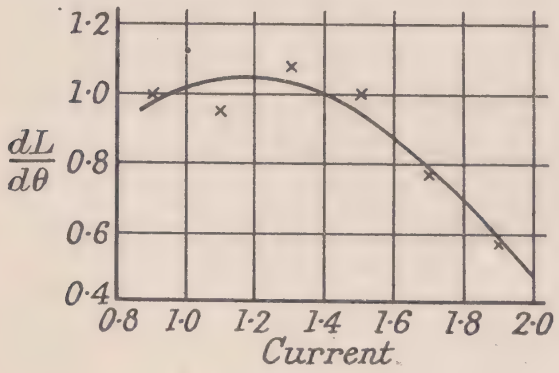


Fig. 43.

The curve so plotted is given in Fig. 43. The selected values are

$$1A,\ \frac{dL}{d\theta} = 0{\cdot}102\ \mu H/\text{degree} = 5{,}850\ \text{cm./radian}.$$

$$2A,\ \frac{dL}{d\theta} = 0{\cdot}048\ \mu H/\text{degree} = 2{,}750\ \text{cm./radian}.$$

At 1 ampere, $I_a = 0{\cdot}1$ and $T = 29{\cdot}2$ dyne cm., and at 2 amperes, $I_a = 0{\cdot}2$ and $T = 55{\cdot}0$ dyne cm.

Work done in increasing the Area of an Electric Circuit.

Two long parallel wires A and B are considered, each of radius r and of length l, separated by a distance x, $l > > x$. They are imagined resistanceless and are supplied with current by a resistanceless generator (Fig. 44). The inductance is (equation 116) $L = 4l \log_e \frac{x}{r}$. The generator is so excited that after a certain time a steady current I_a is circulating. The energy supplied by the generator is $\frac{1}{2}LI_a^2 = 2lI_a^2 \log_e \frac{x}{r}$. If $x = a$ this energy is

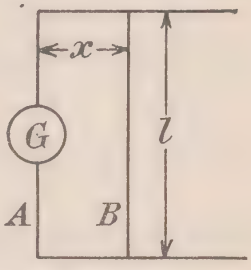


FIG. 44.

$W_a = 2lI_a^2 \log_e \frac{a}{r}$. If now the circuit be broken, x increased to b and the current re-established, the energy supplied by the generator is $W_b = 2lI_a^2 \log_e \frac{b}{r}$. The excess in the position $x = b$ over that in the position $x = a$ is $2lI_a^2 \log_e \frac{b}{a}$.

Now reverting to the position $x = a$ the conductor B is imagined to move from a to b, the current being maintained constant at I_a. At any distance x the force between the conductors (repulsive, tending to increase the area of the circuit) is lHI_a. Since $H = 2I_a/x$ this force is $2lI_a^2/x$. The mechanical work done in moving from a to b is therefore

$$W_m = \int_a^b \frac{2lI_a^2}{x} dx = 2lI_a^2 \log_e \frac{b}{a} \quad . \quad . \quad . \quad (136)$$

During the movement, an e.m.f. is generated in the circuit tending to reduce the current. To maintain the current an equal e.m.f. must be supplied by the generator. This e.m.f. is *not equal to* $lH\frac{dx}{dt}$ but to twice this. If B is the moving conductor an e.m.f. $lH\frac{dx}{dt}$ is generated in it, because it is moving in the flux of A. An equal e.m.f. is generated in A because as B moves its flux cuts A. The total e.m.f. is therefore $2lH\frac{dx}{dt}$.

This result can be obtained another way. At any distance x, the flux linking the circuit is $4lI_a \log_e \frac{x}{r}$. The e.m.f. generated when x changes is $\frac{d}{dt}\left(4lI_a \log_e \frac{x}{r}\right) = \frac{4lI_a}{x}\frac{dx}{dt} = 2lH\frac{dx}{dt}$. It therefore follows that the rate at which energy is being supplied by the generator is $\frac{4lI_a}{x}\frac{dx}{dt} \cdot I_a$. In moving a distance δx in a time δt the energy supplied from the generator is $\frac{4lI_a^2}{x}\frac{dx}{dt}\delta t$ or $\frac{4lI_a^2}{x}\delta x$. The total energy supplied by the generator as the conductor moves from a to b is therefore $\int_a^b \frac{4lI_a^2}{x}dx = 4lI_a^2 \log_e \frac{b}{a}$.

The energy balance is therefore :

Energy supplied by generator $= 4lI_a^2 \log_e \frac{b}{a}$

External work done $= 2lI_a^2 \log_e \frac{b}{a}$

Increase in energy of circuit $= 2lI_a^2 \log_e \frac{b}{a}$

It is interesting to consider this example in connection with the usual method of determining the e.m.f. generated in a conductor when it cuts a magnetic field (see p. 57).

Movement of Electrons in an Electric Field.

The wide use of the cathode ray oscillograph has emphasised the practical aspect of the motion of electrons in electric and magnetic fields. So long as an electron is moving with a velocity well below that of light it may be assumed that its mass is constant. If its charge is q E.S.U. and its mass m grms. then its acceleration in an electric field of strength ε is

$$f = \frac{\varepsilon q}{m} \text{ cm. per sec. per sec} \quad . \quad . \quad . \quad (137)$$

Practical units are based upon the joule as the unit of work, the corresponding unit of force being 10^7 dynes. If therefore the ratio q/m is expressed in coulombs per gramme and the electric field ε in volts per centimetre the acceleration is

$$f = \frac{\varepsilon q}{m} \times 10^7 \text{ cm. per sec. per sec.} \quad . \quad . \quad (138)$$

In moving from rest between two points during which the fall of potential is V the work done is Vq (ergs if E.S.U. are employed, joules if practical units are employed). During the motion the electron gains kinetic energy equal to $\frac{1}{2}mu^2$ where u is the velocity. If practical units are used then

$$Vq \times 10^7 = \tfrac{1}{2}mu^2$$

and

$$u = \sqrt{2 \times 10^7 V \cdot \frac{q}{m}} \quad . \quad . \quad . \quad (139)$$

An application of this is afforded by the following example: "If the accelerating potential used in a cathode ray tube is 1000 volts and the distance between the hot cathode and the accelerating electrode is 10 cm., calculate the maximum velocity of the electron stream on the assumption that the electrons leave the cathode with zero velocity. The ratio of the electron charge to its mass may be taken as $1{\cdot}767 \times 10^8$ coulombs per gramme." * (L.U. Ext. E.M. and M.I., I. 8, 1936, part question.)

From equation 139 the velocity is given as

$$u = \sqrt{2 \times 10^{10} \times 1{\cdot}767 \times 10^8}$$

$$= 1{\cdot}88 \times 10^9 \text{ cm./sec.}$$

This is only just over 6 per cent. of the velocity of light.

If the value of q/m given in the example above be substituted in equation 139 one obtains

$$u = 5{\cdot}94 \times 10^7 \sqrt{V}$$

Electric Deflection of a Cathode Ray.

The usual method of deflecting the cathode beam in an oscillograph is to pass it between two parallel plates between which is a difference of potential. This difference of potential establishes an electric field at right angles to the direction of the beam. During their passage through this field the electrons are given a transverse acceleration and emerge from the field with a transverse velocity u_t in addition to the forward velocity u acquired during acceleration under the main anode potential. The distance traversed in this field is only a small fraction of the total path and it may be considered that the effect of the potential applied between the deflecting plates is to bend the

* A more exact value for this figure has been given for that stated in the original example.

beam through an angle $\tan^{-1}\frac{u_t}{u}$. This angle is usually small and it may usually be assumed that the angle of deflection is u_t/u.

If the length of the deflecting zone is l_d the time taken to pass through it is l_d/u. If practical units are used the acceleration f is $\frac{\varepsilon q}{m} \times 10^7$ cm. per sec. per sec. and hence

$$u_t = \frac{l_d}{u} \cdot \varepsilon \cdot \frac{q}{m} \times 10^7 \text{ cm. per sec.}$$

$$\frac{u_t}{u} = \frac{l_d}{u^2} \cdot \varepsilon \cdot \frac{q}{m} \times 10^7$$

But $$u^2 = 2 \times 10^7 V \frac{q}{m} \text{ (equation 139)}$$

Hence $$\frac{u_t}{u} = \frac{l_d \varepsilon}{2V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

It is seen that the sensitivity is inversely proportional to the accelerating voltage ("gun" voltage); the deflection is proportional to the difference of potential between the deflecting plates.

Again, if the distance between the deflecting plates is d then ε = deflecting voltage/d. The angular sensitivity in radians per volt is $l_d/2Vd$. If the distance between the centre of the deflecting zone and the screen is L then the sensitivity in cm. per volt is $l_d L/2Vd$.

Magnetic Deflection of a Cathode Ray.

When an electric charge moves perpendicularly to a magnetic field there is a force upon it at right angles to the direction of the field and to the direction of motion (p. 57). Since the force is always perpendicular to the direction of motion no work is done upon the charge and its speed is unaltered; only the direction of its motion is changed. A charge q coulombs moving with a velocity of u cm. per second perpendicularly to a field of intensity H is acted upon by a force of $\frac{qHu}{10}$ dynes. Its acceleration perpendicular to its direction of motion is therefore

$$f = \frac{Hu}{10} \cdot \frac{q}{m} \text{ cm. per sec. per sec.} \quad . \quad . \quad (141)$$

The path is curved; if ρ is the radius of curvature then

$$f = \frac{u^2}{\rho}$$

whence
$$\rho = \frac{10u}{H} \cdot \frac{m}{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

If the length of path in the field is l_m then the angle of deflection $\theta = \frac{l_m}{\rho} = \frac{l_m H}{10u} \cdot \frac{q}{m}$. Substituting for u from equation 139 this gives

$$\theta = \frac{Hl_m}{10}\sqrt{\frac{q}{m}\Big/(2 \times 10^7 V)}$$

But $\frac{q}{m} = 1{\cdot}767 \times 10^8$

giving
$$\theta = \frac{0{\cdot}297 Hl_m}{\sqrt{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

CHAPTER VII

HARMONIC AND VECTOR QUANTITIES; SYMMETRICAL COMPONENTS

The direction and magnitude of any vector quantity may be represented by a complex number. Thus $C = r(\cos\theta + j\sin\theta)$ may represent the position of the crank pin of a steam engine (for the moment considered stationary), where r is the length of the crank arm (centre to centre) and θ is the angle between the centre line of the crank arm, and an arbitrary reference line scribed upon the squared end of the crank shaft. Similarly $E = r'(\cos\phi + j\sin\phi)$ may represent the position of the eccentric. C and E are vectors and may be expressed as $C = |C|e^{j\theta}$ and $E = |E|e^{j\theta}$. For convenience it is assumed that the reference line is horizontal in the stationary condition considered. If now the crankshaft be imagined to revolve with a constant velocity ω, subsequent positions of the crank pin and eccentric at any time t, with reference to the centre of the crankshaft and a fixed horizontal line perpendicular to the axis of the crankshaft, become

$$r[\cos(\omega t + \theta) + j\sin(\omega t + \theta)]$$

$$= r(\cos\theta + j\sin\theta)(\cos\omega t + j\sin\omega t) = Ce^{j\omega}$$

and $r'[\cos(\omega t + \phi) + j\sin(\omega t + \phi)] = Ee^{j\omega t}$.

These expressions $Ce^{j\omega t}$ and $Ee^{j\omega t}$ are *rotating vectors.*

The positions of the crank and eccentric define the positions of the piston and slide valve. If the obliquity of the rods be neglected, and for the present purpose it is neglected, uniform rotation of the crankshaft involves simple harmonic motion of piston and valve. The changes of position of these as the crankshaft revolves are given by the construction of Fig. 45. (No importance must be attached to the relative amplitudes and phases of C and E there portrayed. The values have been selected for the purpose of the diagram.) At any time t

the displacements of piston and valve from their mid-positions are given respectively by Oc and Oe.

Analogous displacements of electricity take place when a sinoidal alternating voltage is applied to a branched circuit. The current in each branch will usually be sinoidal, but phase differences as well as differences of magnitude may exist. The current in one branch may be represented by the projection of one rotating vector upon a fixed line, the current in another branch by the projection of another rotating vector upon the same line. Voltages upon various parts of the circuit may be similarly represented.

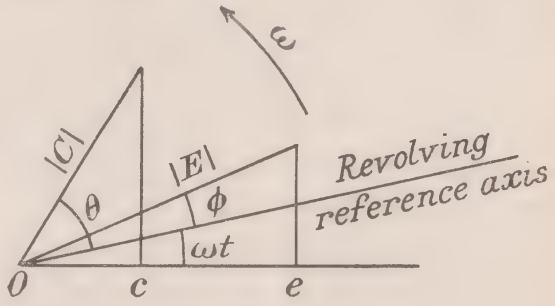


Fig. 45.

The currents themselves are alternating or harmonic quantities, not vector quantities, except in the special sense that they are represented in magnitude and phase by the projections of rotating vectors. Such a representation can be of considerable value if (i) the manipulation of the vectors is easier than the manipulation of the simple harmonic quantities, (ii) there are reciprocal and consistent relations between the vectors and the simple harmonic quantities.

It will be seen in subsequent work that condition (i) is satisfied but that (ii) is only satisfied to a limited, but still very useful, degree. The relations which exist between simple harmonic quantities of the same frequency and their related rotating vectors may be briefly stated (proof follows) :

(*a*) A simple harmonic quantity H is the projection upon a fixed axis of a uniformly rotating vector $\boldsymbol{R}$ of constant length. The amplitude of the harmonic quantity is equal to the modulus of the rotating vector. The frequency of the simple harmonic quantity is $\omega/2\pi$.

(*b*) The sum of a number of simple harmonic quantities, ΣH, is equal to the projection upon the reference axis of the vector sum $\Sigma \boldsymbol{R}$ of the separate rotating vectors $\boldsymbol{R}$ defining the separate harmonic quantities H.

(*c*) The time derivative of a rotating vector $\boldsymbol{R}$, defining a simple harmonic quantity H, is a second rotating vector $\boldsymbol{R}_1$,

defining a simple harmonic quantity H_1, which is the time derivative of H.

(*d*) The product of a *vector* V and a *rotating vector* $\boldsymbol{R}$ is a rotating vector $\boldsymbol{R'}$ which, with a particular interpretation attached to V, may define a new simple harmonic quantity of physical significance.

(*e*) The product of two rotating vectors $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$, defining respectively two harmonic quantities H_a and H_b is a rotating vector of angular velocity 2ω, defining a simple harmonic quantity which is not the product of H_a and H_b and it has no special physical significance.

(*f*) The product of two simple harmonic quantities H_a and H_b, of the same frequency $\omega/2\pi$, is a scalar quantity plus a simple harmonic quantity of double frequency, ω/π. This result cannot, in view of (*e*), be derived by multiplication of the two defining rotating vectors $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$. (A rule is, however, given later [p. 110] whereby the magnitude of the scalar portion of H_aH_b may be abstracted from this artificial product of $\boldsymbol{R}_a$ and $\boldsymbol{R}_b$.)

These relations may be more specially examined.

(*a'*) A rotating vector

$$\boldsymbol{R} = Re^{j\omega t} = |R|(\cos\theta + j\sin\theta)(\cos\omega t + j\sin\omega t)$$
$$= |R|\{\cos(\omega t + \theta) + j\sin(\omega t + \theta)\}$$

It defines a simple harmonic quantity $H = |R|\cos(\omega t + \theta)$ by its projection upon the reference axis.

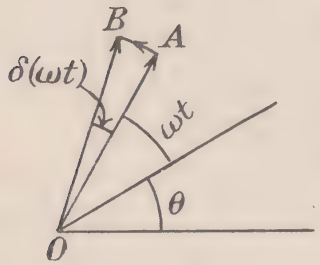


FIG. 46.

(*b'*) The summation relation expressed in (*b*) above follows immediately from equation 1, p. 4.

(*c'*) At any time t, $\boldsymbol{R} = OA$ (Fig. 46).

At a time $t + \delta t$, $\boldsymbol{R} + \boldsymbol{\delta R} = OB$.

$$\boldsymbol{\delta R} = AB\,;\ |OA| = |R|\,;$$
$$|AB| = |OA|\delta(\omega t) = \omega|R|\delta t$$

But AB is a vector $\frac{\pi}{2}$ in advance of OA or $\boldsymbol{R}$.

Hence $$AB = \boldsymbol{\delta R} = j\omega\boldsymbol{R}\,\delta t \text{ or}$$

$$\frac{d\boldsymbol{R}}{dt} = j\omega\boldsymbol{R} \quad . \quad . \quad . \quad . \quad . \quad (144)$$

This result might have been obtained by direct differentiation of $\boldsymbol{R}$ by normal methods, thus

$$\boldsymbol{R} = Re^{j\omega t}$$

$$\frac{d\boldsymbol{R}}{dt} = j\omega Re^{j\omega t} = j\omega\boldsymbol{R}$$

$\boldsymbol{R}$ defines a simple harmonic quantity $H = |R| \cos(\omega t + \theta)$. $j\omega\boldsymbol{R}$ defines a simple harmonic quantity ω times as big as H and $\frac{\pi}{2}$ ahead in phase or $H' = \omega|R| \cos\left(\omega t + \theta + \frac{\pi}{2}\right)$. But

$$\frac{dH}{dt} = -\omega|R| \sin(\omega t + \theta) = \omega|R| \cos\left(\omega t + \theta + \frac{\pi}{2}\right) = H'$$

proving the relation stated in (*c*) above.

The differentiation of a harmonic quantity is equivalent to increasing its magnitude ω times and advancing its phase by $\pi/2$. Differentiation of a rotating vector is equivalent to multiplying the vector by $j\omega$, which has exactly the same effect, namely to increase the magnitude ω times and to advance the phase by $\pi/2$.

(*d'*) If a rotating vector $\boldsymbol{R}$, defining $H(= |R| \cos \overline{\omega t + \theta})$, be multiplied by a vector $V = a + jc = a + j\omega b$ (where $\omega b = c$) a new rotating vector is obtained

$$V\boldsymbol{R} = (a + j\omega b)\boldsymbol{R} = a\boldsymbol{R} + b\frac{d\boldsymbol{R}}{dt}$$

The corresponding harmonic quantity is (by *c'*)

$$aH + b\frac{dH}{dt} = a|R| \cos(\omega t + \theta) + c|R| \cos\left(\omega t + \theta + \frac{\pi}{2}\right) \quad . \quad . \quad (145)$$

Although a vector, V is often called a *vector operator* or *complex operator* when the term for $\boldsymbol{R}$ is shortened to vector.

(*e'*) It is seen that the product of two rotating vectors $\boldsymbol{R}_a = R_a e^{j\omega t}$ and $\boldsymbol{R}_b = R_b e^{j\omega t}$ is $\boldsymbol{R}_a\boldsymbol{R}_b = R_a R_b e^{2j\omega t}$, a vector rotating at an angular velocity 2ω. No particular significance is attached in engineering work to this relation.

(*f'*) The product of two harmonic quantities

$H_a = |R_a| \cos(\omega t + \theta)$ and $H_b = |R_b| \cos(\omega t + \phi)$ is

$$H_a H_b = \frac{|R_a||R_b|}{2}\{\cos(2\omega t + \theta + \phi) + \cos(\theta - \phi)\} \quad (146)$$

It is seen that this is the sum of a double frequency simple

harmonic quantity and a scalar quantity. Commonly only the average value of the product is required and this is given by

$$\text{Average value of } H_a H_b = \frac{|R_a||R_b|}{2} \cos(\theta - \phi) \qquad (147)$$

Vectorial Representation of a Wave Motion.

The essential characteristic of a wave is the harmonic variation of some quantity with respect to two independent variables. In the more familiar instances these independent variables are time and distance. Thus in water waves, at any particular instant, there is an harmonic variation of level of the surface in space, and again, at any particular place there is an harmonic variation of level in time. The "distribution" of the surface level moves with a definite velocity. The wave is perhaps best considered from this point of view. At the point $x = 0$ the motion may be represented by the rotating vector

$$\boldsymbol{R_0} = Re^{j\omega t} = Re^{j2\pi ft}$$

where f is the frequency or the number of vibrations per *second*. The phase at any time t is ωt. This "motion" may be imagined moving in the direction of x with a velocity u. At the point x the motion is the same as at $x = 0$ except that it occurs later by the time taken for the motion to travel the distance x, namely $\frac{x}{u}$. Therefore the rotating vector representing the motion at x is

$$\boldsymbol{R_x} = Re^{j\omega\left(t-\frac{x}{u}\right)}$$

At a distance λ defined by

$$\frac{\omega\lambda}{u} = 2\pi$$

the motion is exactly the same in magnitude and phase as at $x = 0$

$$\lambda = \frac{2\pi u}{\omega} = \frac{u}{f} \text{ or } u = f\lambda$$

λ is the wave-length, or the distance in which one complete

wave is situated. The number of waves in a centimetre, or the *wave number* is

$$n = \frac{1}{\lambda} = \frac{f}{u}$$

whence

$$u = \frac{f}{n}$$

Making this substitution gives

$$\boldsymbol{R}_x = Re^{j\omega\left(t-\frac{nx}{f}\right)} = Re^{j2\pi(ft-nx)}$$

[The equation just given is a particular case of the general equation to a travelling wave, or disturbance, namely $y = \phi(x - ut)$. It may be helpful to read the discussion of equation 648 on p. 266.]

Application to Alternating Currents.

The rotating vectors so far considered have been of magnitude equal to the maximum values of the harmonic quantities they define. Engineers almost always draw the vectors to represent by their lengths the R.M.S. values of voltages and currents. When so drawn the projections upon the reference axis must be multiplied by $\sqrt{2}$ to give instantaneous values. The relations existing between the rotating vectors, vectors, and harmonic quantities involved in alternating currents follow immediately from the work of the preceding paragraphs. Corresponding to a simple harmonic current of R.M.S. value $|I|$ and of phase angle θ the vector is $I = |I|\varepsilon^{j\theta}$ and the rotating vector $I\varepsilon^{j\omega t} = |I|\varepsilon^{j\omega t}\varepsilon^{j\theta}$. These relations are shown in Fig. 46A. If this current flows through a resistance R the instantaneous value of the potential difference across the resistance is $v = iR$, the corresponding vector and rotating vector being respectively $V = IR$ and $Ve^{j\omega t} = Ie^{j\omega t}R$.

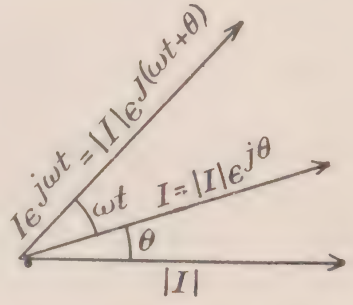


FIG. 46A.

The potential difference required to maintain an alternating current, of instantaneous value i, through an inductance L is $v = L\frac{di}{dt}$. The corresponding vector is $V = j\omega LI$ and the corresponding rotating vector is $Ve^{j\omega t} = j\omega LIe^{j\omega t}$.

An alternating potential of instantaneous value v applied to a capacitance C results in an alternating current of instantaneous value $i = C\frac{dv}{dt}$. In vector notation $I = j\omega CV$ and $Ie^{j\omega t} = j\omega CVe^{j\omega t}$. It follows that $V = I/j\omega C = -jI/\omega C$.

When employing vector notation it is often not necessary to deal with the rotating vectors; usually all the required relations are given by consideration of the vectors. One must not, however, lose sight of the fact that the actual relevant quantities are harmonic and are given by the projections of the corresponding rotating vectors.

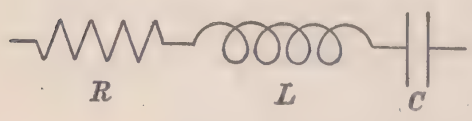


Fig. 47.

The general relations are illustrated by a simple example. A current I (vector) flows through the circuit of Fig. 47, consisting of a resistance R in series with an inductance L and a capacitance C, then the vector potential is

$$V = IR + j\omega LI + \frac{I}{j\omega C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

or $$V = I\left\{R + j\left(\omega L - \frac{1}{\omega C}\right)\right\} = I(R + jX) = IZ$$

where $X = \omega L - \frac{1}{\omega C}$ and $Z = R + jX$.

Selecting the horizontal direction as that of the vector I, the relations are shown in Fig. 48. R and X are scalar quantities, R the resistance and X the *reactance*. $R + jX = Z$ is a complex quantity, the *complex impedance* or *vector impedance*. $|Z| = \sqrt{R^2 + X^2}$ is the *impedance* (a scalar). Different notations are commonly employed in expressing currents and voltages. When working in vectors it is convenient to use the symbolism here adopted; V is a vector voltage, $|V|$ its scalar magnitude. When, however, other than vector methods are being employed, it is convenient to let V stand for the scalar

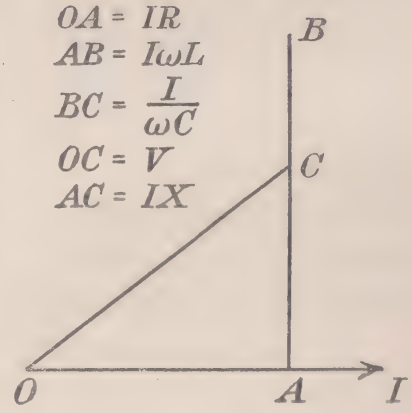


Fig. 48.

magnitude of the voltage. Then if it is necessary to specify the vector it may be represented by $\dot{V}$.

Vector Admittance.

The vector equation $V = IZ$ may be written as $I = VY$ where $Y = 1/Z$. Y is the *vector admittance.*

$$Z = |Z|(\cos\phi + j\sin\phi) = R + jX$$

$$Y = \frac{1}{Z} = \frac{1}{|Z|(\cos\phi + j\sin\phi)} = |Y|(\cos\phi - j\sin\phi) = G - jB$$

G is the *conductance*, B the *susceptance*, both scalar quantities

$$\tan\phi = \frac{X}{R} = \frac{B}{G}$$

Inversion and Vector Loci.

The vector impedance Z expressed in ohms (Ω) is represented in Fig. 49, together with the vector admittance Y expressed in mhos ($\mho$).

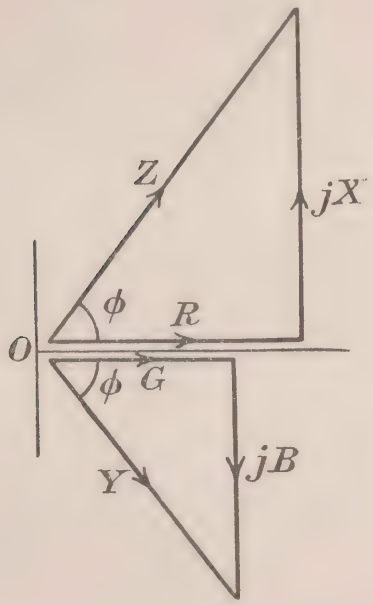


Fig. 49.

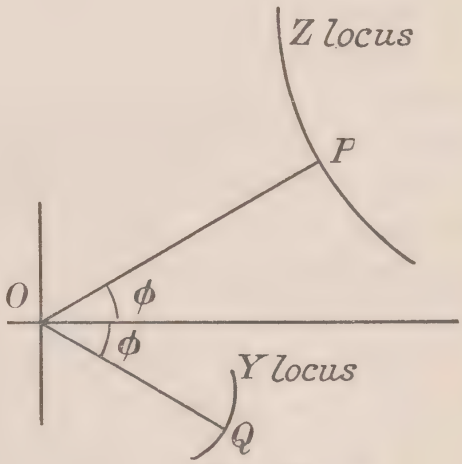


Fig. 50.

Cases arise, for instance in the operation of an alternating current motor, when the effective vector impedance varies in a definite manner throughout the range of operation. A vector impedance locus may be drawn and also the corresponding vector admittance locus. Such a possible pair of loci are shown in Fig. 50. Corresponding to any point P on the Z locus there is a point Q on the Y locus; $OP \times OQ$ = constant (the constant of inversion). P and Q are termed *in-*

verse points. It is generally more convenient to plot the admittance as if its value were $G + jB$ instead of $G - jB$. The point Q is then reflected to Q' (Fig. 51) on the line OP.

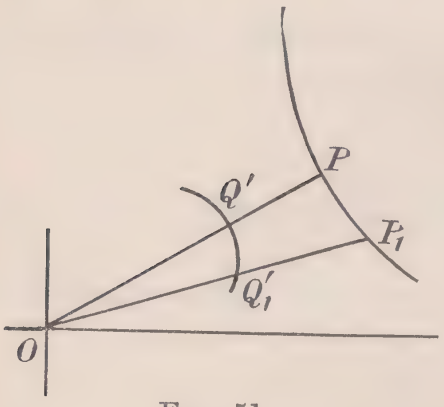


Fig. 51.

Q' instead of Q is then treated as the inverse of P. To obtain admittance from impedance then theoretically requires reflection of the derived locus across the reference line. It is rarely necessary actually to perform the reflection since the shape of the locus is unaffected. When actual values of impedance or admittance are required the distances above the reference axis are reckoned as positive imaginary for the Z locus and negative imaginary for the Y locus. The geometrical advantage obtained by O, Q' and P being collinear is obvious.

Inversion of a Straight Line.

BAC (Fig. 52) is the straight line, A being the foot of the perpendicular dropped from the pole O. The inverse point of A is A' where $OA \,.\, OA' = k$, the constant of inversion. If B' is the inverse of B then $OB' \,.\, OB = k$. Hence $OA'/OB' = OB/OA$. It therefore follows that the two triangles OAB and $OB'A'$ are similar and therefore the angle $OB'A'$ is a right angle. Hence B', and therefore the inverse of any point on the line BAC, is on the circle having OA' as diameter.

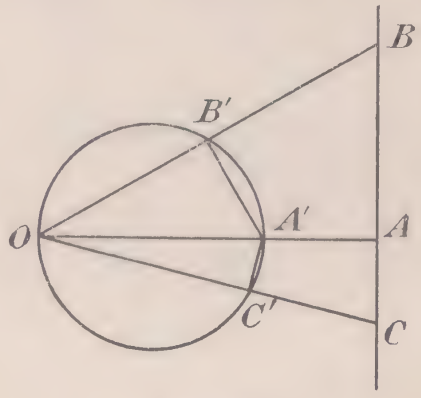


Fig. 52.

Inversion of a Circle.

In general a circle inverts to another circle. The case just treated is a particular example of this since a straight line is a circle of infinite radius. The circle is APB having its centre at C_1, A and B are the diametral intercepts of the line OC_1, O being the pole. A' and B' are the inverse points of A and B. $OA \,.\, OA' = k = OB \,.\, OB'$. A circle is described on $A'B'$. It

is required to prove that if P' is the inverse of P then it lies on the circle having $A'B'$ as diameter. $OP \,.\, OP' = k$. From the relations given it is clear that the triangles OBP and $OP'B'$ are similar. Hence $\angle OPB = \angle OB'P'$. Again, the triangles

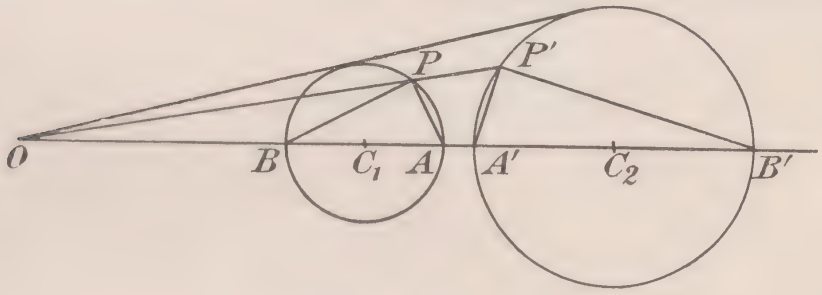


FIG. 53.

OAP and $OP'A'$ are similar and therefore $\angle OPA = \angle OA'P'$. Hence $\angle BPA = \angle B'P'A' = \frac{\pi}{2}$ and therefore P' is on the circle having $A'B'$ as diameter. It follows that one pair of common tangents intersect at O.

Example.

A circuit consists of two branches in parallel. The first branch is a variable resistance R in series with a reactance of 10 ohms. The second branch is a resistance of 10 ohms in series with a reactance of 5 ohms. Draw the current locus when 100 volts alternating is applied and from it determine the value of the current when $R = 5$ ohms.

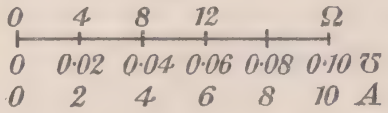
$Z_1 = R + j10$; $Z_2 = 10 + j5$.

Convenient scales are chosen. The locus of Z_1 is a straight line AP parallel to the real axis (Fig. 54). Inverting this with respect to the pole O gives the admittance locus Y_1. This is a circle, the diameter OB being perpendicular to AP. When R is zero, $Z_1 = j10$, represented by OA, Y_1 is $-j0{\cdot}1$, represented by OB. The circle is described upon OB as diameter. If Z_1 be represented by OP, the admittance of branch 1 is represented by OP' if it be remembered that the component $P'N$ (the susceptance) must be considered negative (since the true inverse of P is the reflection of P' in Oa).

The admittance of branch 2 is

$$Y_2 = \frac{1}{10 + j5} = \frac{10 - j5}{125} = 0{\cdot}08 - j0{\cdot}04$$

The total admittance is therefore obtained by adding $0{\cdot}08 - j0{\cdot}04$ to OP'. This is conveniently done by selecting a new pole from which the total admittance may be measured. Remembering that negative susceptances are plotted upwards to avoid reflection this means selecting a pole O' a distance



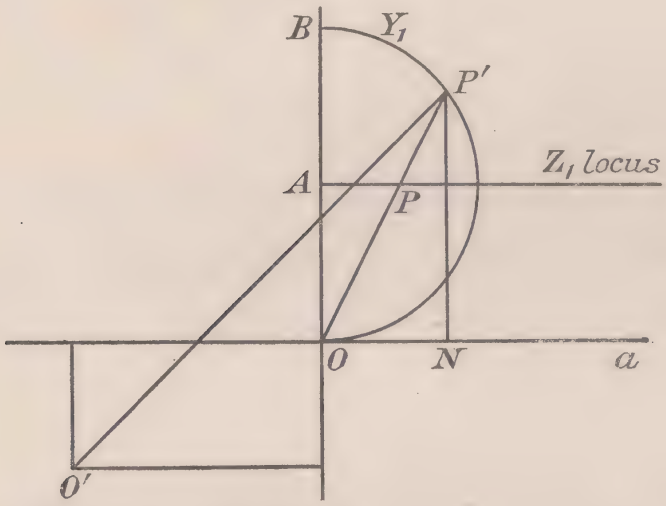


Fig. 54.

0·08 to the left and 0·04 below O. This has been done in Fig. 54. In the case shown, P is selected for the particular case of $R = 5$ ohms. The resultant admittance $O'P'$ is 0·17 mho and the current is 17 amperes. The angle of *lag* is 45°.

Resonant Circuit.

Another interesting example of this method is afforded by the series resonant circuit. The circuit of Fig. 47 may become resonant by variation of the magnitude of either or both of the reactive components L or C or by variation of frequency. The former case will be considered first. The vector impedance $Z = R + jX$ is represented in Fig. 55 by the line XX. Inverting XX with respect to O the circular admittance locus is

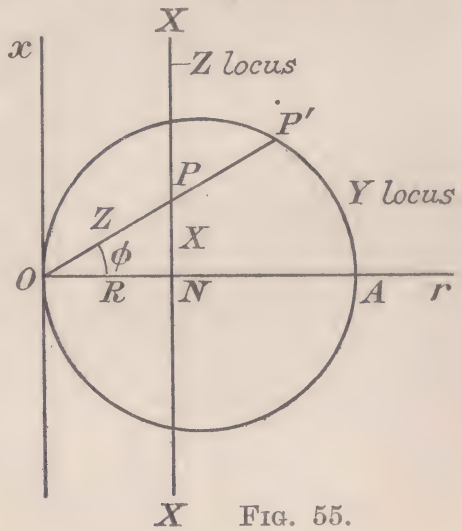


Fig. 55.

obtained. The scale is given by the fact that the admittance represented by $OA = 1/R$. Corresponding to any value of X represented by NP the admittance is represented by OP', the angle of *lag* being ϕ.

When the individual reactances ωL and $\frac{1}{\omega C}$, being nearly equal, are each much greater than R the effect of frequency variation becomes important. The reactance $X = \omega L - \frac{1}{\omega C}$ is zero when $\omega = 1/\sqrt{LC}$. Denoting this value of ω by ω_0, a slightly different frequency may be denoted by $\omega = \omega_0(1 + h)$ where h is small. Then $X = \omega_0 L(1 + h) - \frac{1}{\omega_0 C(1 + h)}$ but $\frac{1}{\omega_0 C} = \omega_0 L$; hence $X = \omega_0 L\left(1 + h - \frac{1}{1 + h}\right)$ or approximately

$$X = 2h\omega_0 L \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

For the type of circuit used in wireless work, deliberately designed to work at or near the resonant frequency, $\omega_0 L >> R$ and equation 149 is almost exact. The length NP may then represent to scale the frequency divergence from the resonant value. At any value of h the admittance is $1/\sqrt{R^2 + (2h\omega_0 L)^2}$. The current is reduced to $1/\sqrt{2}$ of its resonant value when $R^2 = (2h\omega_0 L)^2$ or when $h = \pm \frac{R}{2\omega_0 L}$. If these two values of h are denoted by h_1 and h_2 then

$$\pi(h_1 - h_2) = \frac{\pi R}{\omega_0 L} = \frac{R}{2Lf} \quad . \quad . \quad . \quad . \quad (150)$$

This quantity, the *logarithmic decrement*, is of importance in wireless work.

Coupled Circuits.

For a simple alternating circuit to which a potential E is applied the vector equation is (from equation 148), $E = I(R + jX)$. If this circuit be coupled to a second by a mutual inductance M, the current in the second circuit being I_2, then the equation for the first circuit becomes

$$E_1 = I_1(R_1 + jX_1) + I_2 j\omega M \quad . \quad . \quad . \quad (151)$$

If it be assumed that the second circuit contains no e.m.f. other

than that induced from the first circuit, the similar equation for the second circuit is

$$O = I_2(R_2 + jX_2) + I_1 j\omega M \quad . \quad . \quad . \quad (152)$$

From these two equations the general conditions of the two circuits may be determined. Thus, if I_2 be eliminated from equation 151 by substitution from equation 152 one obtains

$$E_1 = I_1(R_1 + jX_1) + I_1\frac{\omega^2M^2}{R_2 + jX_2} \quad . \quad . \quad (153)$$

or

$$E_1 = I_1\left\{Z_1 + \frac{\omega^2M^2}{Z_2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

The apparent impedance of the primary circuit is

$$Z' = Z_1 + \frac{\omega^2M^2}{Z_2} \quad . \quad . \quad . \quad . \quad . \quad (155)$$

This again may be written

$$Z' = R_1 + jX_1 + \frac{\omega^2M^2}{R_2 + jX_2}$$

or

$$Z' = R_1 + jX_1 + \frac{\omega^2M^2}{R_2^2 + X_2^2}(R_2 - jX_2) \quad . \quad (156)$$

indicating that the apparent resistance is *increased* by $\frac{\omega^2M^2}{Z_2^2} \cdot R_2$ and the apparent reactance is *decreased* by $\frac{\omega^2M^2}{Z_2^2} \cdot X_2$. Similarly by eliminating I_1 the secondary current is given as

$$I_2 = -\frac{j\omega ME_1}{Z_1Z_2 + \omega^2M^2} \quad . \quad . \quad . \quad . \quad (157)$$

Example. A coil having a resistance of 1 ohm and an inductance of 3 microhenries is coupled by a mutual inductance of 2 microhenries with a resonant circuit of 5 ohms resistance, 50 microhenries inductance and 800 micro-microfarads capacitance. Draw the vector locus of the impedance of the first coil as the frequency is varied through resonance (L.U. Ext. E.M. and M.I., II. 5, 1937, part question).

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{50 \times 10^{-6} \times 800 \times 10^{-12}} = 5 \times 10^6$$

The apparent impedance of the first coil is $Z_1 + \frac{\omega^2M^2}{Z_2}$ (equation 155). The second term of this expression is obtained by

inverting Z_2 with an inversion factor ω^2M^2. Over the range required it may be assumed that ω^2M^2 is constant and equal to $\omega_0{}^2M^2$ or 100. When $\omega = \omega_0$ then $Z_2 = 5$ giving $(\omega^2M^2/Z_2)_0 = 20$. Again $\omega_0L_1 = 15$ giving $Z_1 = 1 + j15$. The construction is therefore clear. The locus of Z_2 is first drawn; it is a line parallel to Ox at a distance of 5 ohms to scale (Fig. 56). This is inverted, giving the circle of diameter $(\omega^2M^2/Z_2)_0 = 20$ ohms to scale. The equivalent of reflection must be performed. This is conveniently done by marking a frequency scale (or rather a scale of h) along the locus of Z_2, positive values of h being plotted downwards.

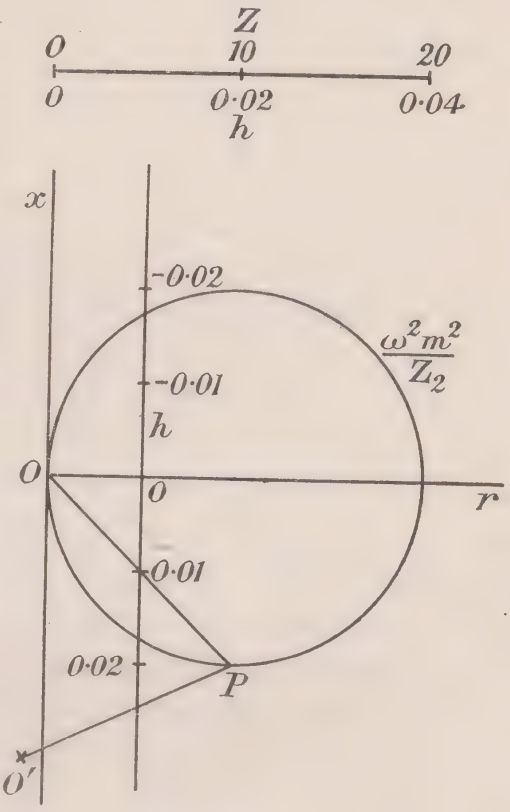


FIG. 56.

$$h = \frac{X_2}{2\omega_0L_2} = \frac{X_2}{500} \text{ whence the scale is given.}$$

Thus corresponding to the value of $h = 0{\cdot}01$, ω^2M^2/Z_2 is represented by OP. To this must be added Z_1, which may be assumed constant. This is done by taking a new pole O', 1 ohm to scale to the left of O and 15 ohms to scale below. Then $Z' = Z_1 + \dfrac{\omega^2M^2}{Z_2}$ is represented by $O'P$.

The above examples introducing inversion have been chosen to illustrate the application of vector methods because they involve a specific principle in themselves. Most commonly the application of vector methods provides nothing more than an orderly manner of proceeding through the masses of arithmetic which many alternating-current problems involve. This is perhaps their greatest and most common value. The difference in analytical method between dealing with vectors and with harmonic quantities is more apparent than real and with practice it becomes easy to switch from complex to trigonometrical methods as occasion suggests.

Power in an Alternating Current Circuit.

Assuming that the voltage and current are sinoidal the instantaneous values may be expressed as

$$v = \sqrt{2}V \cos \omega t \text{ and } i = \sqrt{2}I \cos (\omega t - \phi)$$

where V and I are the R.M.S. values. The instantaneous power is therefore

$$vi = VI\{\cos(2\omega t - \phi) + \cos \phi\} \quad . \quad . \quad . \quad (158)$$

Over a long time (compared with the period of the A.C.) the average value is $VI \cos \phi$. There is, however, a double-frequency oscillating component of power $VI \cos (2\omega t - \phi)$ and, unless $\cos \phi = 1$, there are intervals during which the circuit is returning energy to the source of supply. Often only the average power is of interest; the watts may then be expressed as the voltage multiplied by the in-phase component of the current. Although it is clear that vector methods are inapplicable to the determination of power (see p. 99, e' and f') this average component may be determined from the expressions for the vectors. If the respective voltage and current vectors be $a + jb$ and $c + jd$, it is readily shown that if ϕ be the angle between them that

$$\cos \phi = \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}} \text{ and } \sin \phi = \frac{ad - bc}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}.$$

Since the magnitudes of the vectors are respectively $\sqrt{a^2 + b^2}$ and $\sqrt{c^2 + d^2}$ it follows that the average watts are $ac + bd$ and the reactive volt amperes, $VI \sin \phi$, are $ad - bc$. The product of $a + jb$ and $c + jd$ is meaningless.

Polyphase Systems.

A polyphase generator has a series of exactly similar windings regularly displaced in space the turns of which are cut by the rotating magnetic field. In these separate phase windings e.m.f.s are generated which are exactly similar except that there are equal phase displacements between them. Assuming that the voltages are sinoidal and that there are n phases, the displacement between two adjacent phases is $2\pi/n$. If the sequence of the phase vectors be V_a, V_b, V_c, V_d, etc., then

$$V_a = V_b e^{j\frac{2\pi}{n}}, \; V_b = V_c e^{j\frac{2\pi}{n}} \; \ldots \text{ etc.} \quad . \quad . \quad (159)$$

Since n is always a whole number it follows that the sum of all the vectors V_a, V_b, V_c, V_d, etc. $= 0$ since they form a regular closed polygon of n sides. From this it follows (a) that all the phase windings may be connected in order in series if desired (mesh), (b) that if the similar ends of all the phase windings be joined together (star—Fig. 57) and each phase be similarly loaded by star-connected equal inpedances, the current in the common return lead $O'O$ is zero and this lead may be omitted. The return lead of one phase may be considered as the leads to the other $n - 1$ phases.*

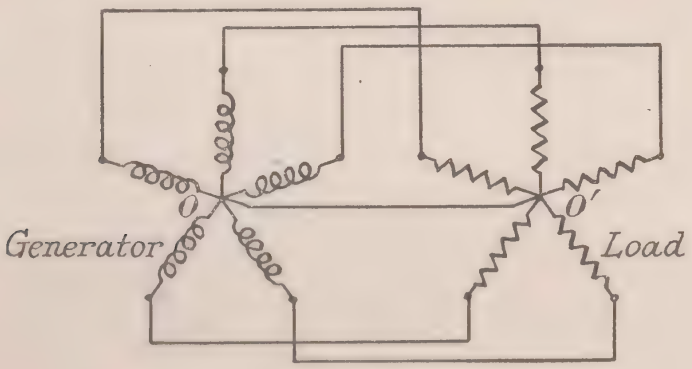


FIG. 57.

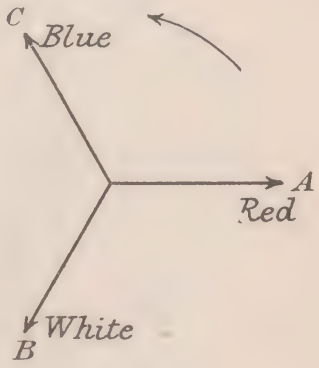


FIG. 58.

Though other cases are of importance, attention will only be directed to the simplest and commonest polyphase system—the 3-phase.

The conventional sequence of vectors in the 3-phase system is illustrated in Fig. 58. It is A, B, C or Red, White (or Yellow), Blue. The colour distinction is useful in wiring, the distinction by letter is convenient in analysis

$$V_A = V_B e^{j\frac{2\pi}{3}} = V_C e^{j\frac{4\pi}{3}} \quad . \quad . \quad . \quad . \quad (160)$$

or

$$V_A = aV_B = a^2 V_C \quad . \quad . \quad . \quad . \quad . \quad (161)$$

where $a = e^{j\frac{2\pi}{3}}$.

a is a unit operator which advances the phase of a vector upon which it operates by $2\pi/3$. Expressed as a complex quantity it is

$$a = e^{j\frac{2\pi}{3}} = \left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

* This is not true for the so-called 2-phase system. This system is irregular and is really one-half of a 4-phase system.

The values of powers of this vector are conveniently represented in the diagrams Figs. 59A and 59B. The following relations are of interest:

$a^3 = 1, j = a^{\frac{3}{4}}, a = a^{-2}, 1 + a = a^{\frac{1}{2}} = -a^2, 1 + a + a^2 = 0.$

Equation 161 is more conveniently written as

$$V_b = a^2 V_a, \quad V_c = a V_a \quad . \quad . \quad . \quad . \quad (162)$$

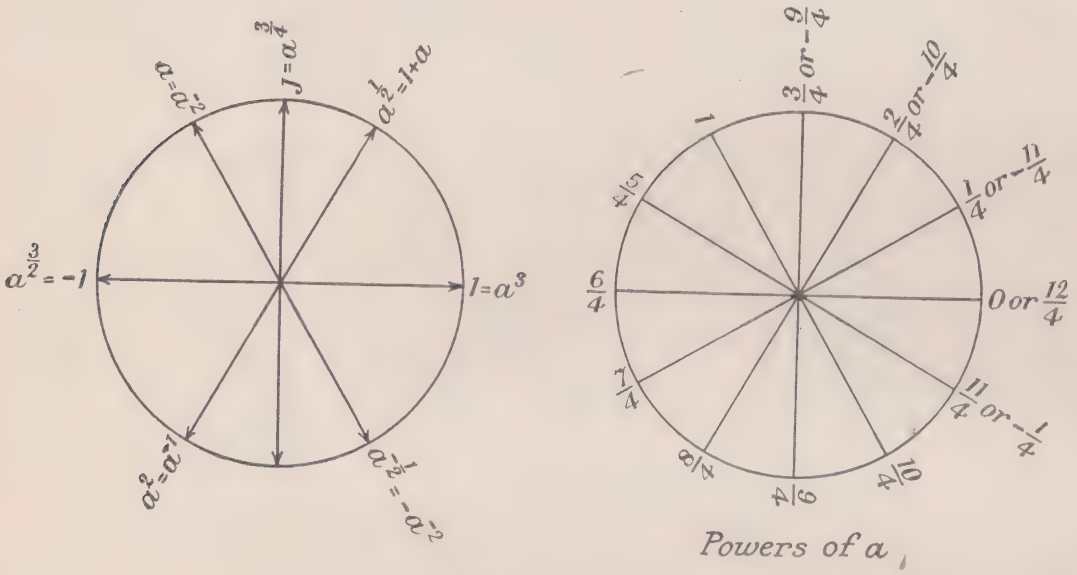


FIG. 59.

Balanced and Unbalanced 3-phase Systems.

The voltage vectors of a 3-phase system have so far been considered as equal in magnitude and differing in phase by $2\pi/3$. The phase windings may be connected in mesh when there are only 3 lines. Loading can then only be effected by connection of admittances between the lines. If the phase windings are connected in star there may be a *neutral* wire joined to the common point of the three phases. If there is not a neutral wire external conditions cannot differ from those of the mesh connection. If there is a neutral wire loading may be effected by the connection of admittances either between the lines or between lines and neutral (the term line is here reserved for the connection to the free end of a phase). If a symmetrical distribution of impedances is employed each phase is similarly and equally loaded. Such a system is said to be balanced; otherwise it is unbalanced. One simple but important fact emerges from consideration of a 3-line system, namely, the sum of the currents in the three lines must be zero

at every instant, whether the system be balanced or unbalanced. If a neutral wire be used with a star-connected system, there may be a current in it if the system is unbalanced. In this case the sum of the 3-line currents is not zero. In every case, however, the sum of the voltages between lines must be zero.

Unbalance may consist in either difference of magnitudes of the vectors or inequality of the phase angles between them, or both. The generated phase voltages are likely to be equal. The connection of unequal impedances may result in current vectors of very different magnitudes and the angles between them may be very different from $2\pi/3$. Such conditions will

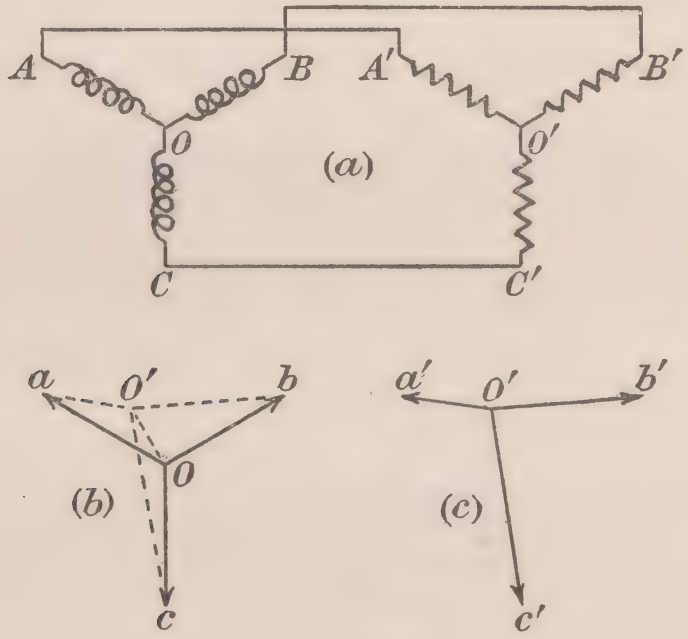


FIG. 60.

produce slight unbalance of the generator terminal voltages owing to the different drops in the phase windings. There may be a much greater degree of unbalance of other voltage vectors. Thus if a star-connected generator supplies current to three very different resistances connected in star (Fig. 60*a*), though the phase voltages in OA, OB and OC may be practically balanced and represented by Oa, Ob, and Oc, the voltages on the resistances may be represented by three vectors such as $O'a'$, $O'b'$ and $O'c'$ (Fig. 60*c*). If O is at zero potential O' has an alternating potential represented by the vector OO' (Fig. 60*b*).

Power in a 3-phase Balanced Circuit.

If a 3-phase circuit is balanced the voltages and currents in the 3 phases may be represented by

$$v_a = \sqrt{2}V \cos \omega t\,; \quad v_b = \sqrt{2}V \cos\left(\omega t - \frac{2\pi}{3}\right);$$

$$v_c = \sqrt{2}V \cos\left(\omega t - \frac{4\pi}{3}\right)$$

$$i_a = \sqrt{2}I \cos(\omega t - \phi)\,; \quad i_b = \sqrt{2}I \cos\left(\omega t - \phi - \frac{2\pi}{3}\right);$$

$$i_c = \sqrt{2}I \cos\left(\omega t - \phi - \frac{4\pi}{3}\right)$$

and the total power is

$$v_a i_a + v_b i_b + v_c i_c = VI\left\{3 \cos \phi + \cos(2\omega t - \phi) + \cos\left(2\omega t - \phi - \frac{4\pi}{3}\right) + \cos\left(2\omega t - \phi - \frac{8\pi}{3}\right)\right\}.$$

But

$$\cos(2\omega t - \phi) + \cos\left(2\omega t - \phi - \frac{4\pi}{3}\right) + \cos\left(2\omega t - \phi - \frac{8\pi}{3}\right) = 0$$

The sum of the oscillating power components is zero and the total power at every instant is $3VI \cos \phi$.

Symmetrical Components.

The general analysis of 3-phase unbalanced circuits is not particularly difficult but is extremely complicated. For this reason it is often not easy to follow the general implications of the results obtained. A greater semblance of order may be introduced into the analysis by the use of what are known as *symmetrical components*. Though the basis of this method is not new it is only within recent years that the processes involved have been systematised with generally accepted conventions.

The most general system of 3 vectors may be represented by F_a, F_b and F_c (Fig. 61). Such a system requires six variables to be defined, namely the modulus and argument of each. These vectors may be considered as the sum of three other systems (Fig. 62), (1) a system of balanced vectors F_{a1}, F_{b1}, F_{c1} having the same sequence as F_a, F_b and F_c, termed the positive sequence system, (2) a system of balanced vectors

F_{a2}, F_{b2}, F_{c2}, of reversed sequence, termed the negative sequence system, and (0) three equal vectors in phase F_{a0}, F_{b0} and F_{c0}, termed the zero sequence system. Two variables are provided by (1). Since the sequence is reversed for (2) another two variables are thereby provided and two variables

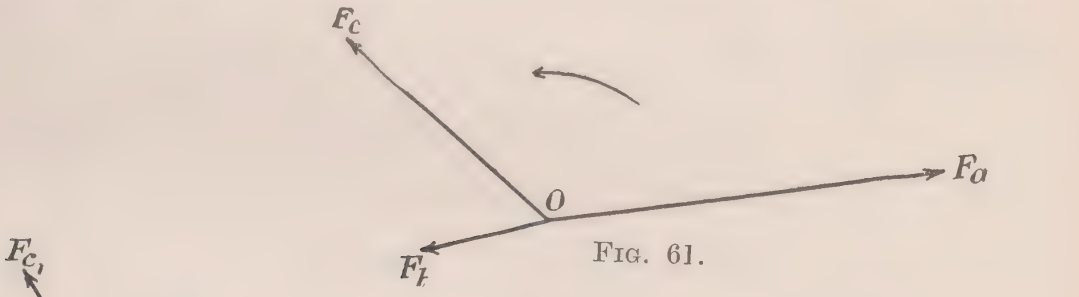


FIG. 61.

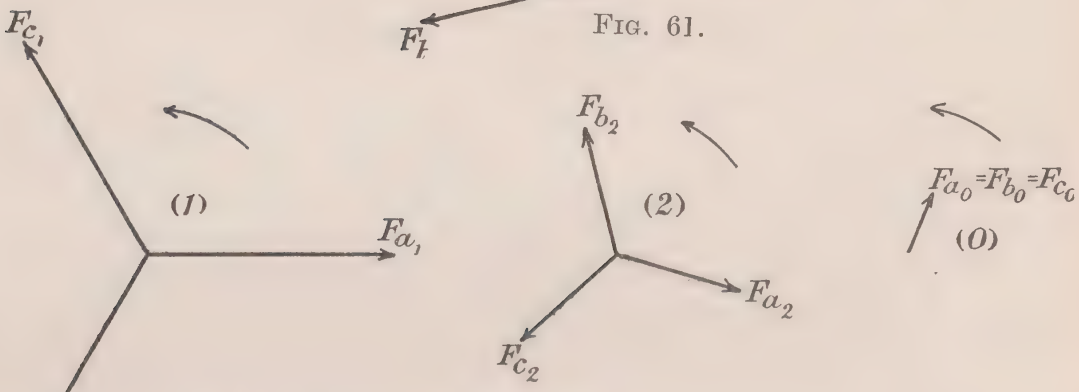


FIG. 62.

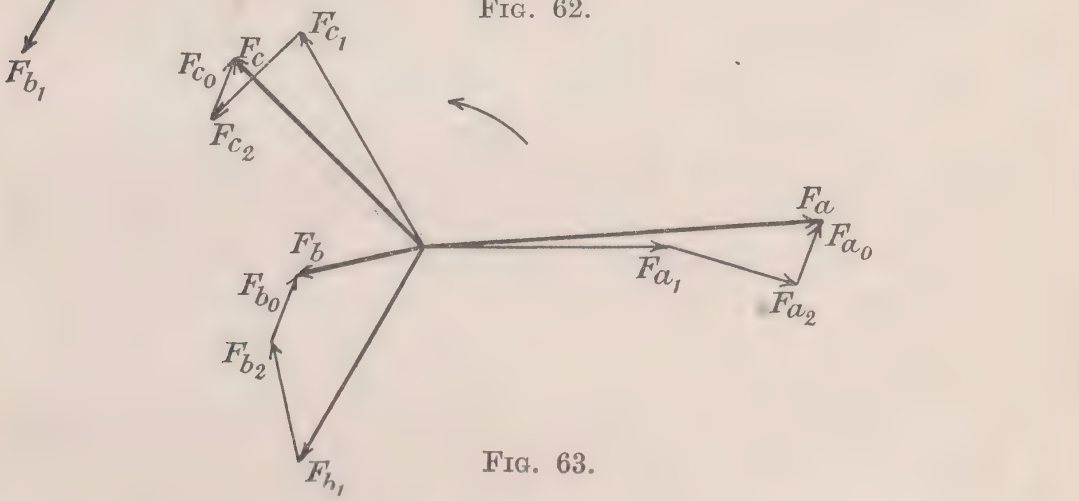


FIG. 63.

are provided by (0). Fig. 63 shows the sum of the three "symmetrical" systems giving the original unbalanced system. For the positive sequence system (from equation 162)

$$F_{b1} = a^2 F_{a1},\ F_{c1} = aF_{a1}$$

For the negative sequence system

$$F_{b2} = aF_{a2},\ F_{c2} = a^2 F_{a2}$$

For the zero sequence system

$$F_{a0} = F_{b0} = F_{c0}$$

Then $F_a = F_{a0} + F_{a1} + F_{a2}$ . . . . . (163)

$$F_b = F_{b0} + F_{b1} + F_{b2} = F_{a0} + a^2F_{a1} + aF_{a2} \quad . \quad (164)$$

$$F_c = F_{c0} + F_{c1} + F_{c2} = F_{a0} + aF_{a1} + a^2F_{a2} \quad . \quad (165)$$

Adding 163, 164 and 165 gives

$$F_{a0} = (F_a + F_b + F_c)/3 \quad . \quad . \quad . \quad (166)$$

Adding 163, a times 164, and a^2 times 165 gives

$$F_{a1} = (F_a + aF_b + a^2F_c)/3 \quad . \quad . \quad . \quad (167)$$

Adding 163, a^2 times 164, and a times 165 gives

$$F_{a2} = (F_a + a^2F_b + aF_c)/3 \quad . \quad . \quad . \quad (168)$$

The case shown in Fig. 61 is taken as an example $F_a = 64{\cdot}0 + j4{\cdot}0$; $F_b = -20{\cdot}8 - j4{\cdot}3$; $F_c = -31{\cdot}2 + j30{\cdot}3$. Substituting these values in equations 166, 167 and 168 and putting $a = -0{\cdot}5 + j0{\cdot}866$ and $a^2 = -0{\cdot}5 - j0{\cdot}866$ gives $F_{a0} = 4 + j10$, $F_{a1} = 40$ and $F_{a2} = 20 - j6$ the values for which these figures were drawn.

The case just considered is that of a system for which the sum of the three vectors is not zero. There are two distinct types of unbalanced 3-phase vector systems, those in which the sum of the three vectors is zero and those in which it is not. The resultant, if there is one, is called the residue. In a 3-line power system the sum of the 3 line currents is zero, or there is no residue ; there is no zero sequence current. It is usual to earth the neutral point of the generator. Should an earth occur on the load (at a point which would not normally be at earth potential), the sum of the 3 line currents will no longer be zero ; in effect the power system has become 4-wire. The detection of a zero sequence current may therefore be used as indication of the development of a fault.

The vectors representing the potentials of the 3 lines V_a, V_b and V_c may, or may not, have a residue. But the differences of potentials between lines V_{ab}, V_{bc}, V_{ca} can have no residue (Fig. 64). This point is emphasised in the following paragraph.

Derived Systems.

The system of vectors V_{ab}, V_{bc} and V_{ca} of Fig. 64 is derived from the system V_a, V_b and V_c. The properties of the derived

systems will be considered more closely. From equations 163, 164 and 165

$$F_{ab} = F_a - F_b = F_{a1}(1 - a^2) + F_{a2}(1 - a)$$
$$= \sqrt{3}a^{\frac{1}{6}}F_{a1} + \sqrt{3}a^{-\frac{1}{6}}F_{a2} \quad . \quad . \quad (169)$$

$$F_{bc} = F_b - F_c = F_{a1}(a^2 - a) + F_{a2}(a - a^2)$$
$$= \sqrt{3}a^{-\frac{3}{6}}F_{a1} + \sqrt{3}a^{\frac{3}{6}}F_{a2} \quad . \quad . \quad (170)$$

$$F_{ca} = F_c - F_a = F_{a1}(a - 1) + F_{a2}(a^2 - 1)$$
$$= \sqrt{3}a^{-\frac{7}{6}}F_{a1} + \sqrt{3}a^{\frac{7}{6}}F_{a2} \quad . \quad . \quad (171)$$

The derived system therefore contains no zero sequence component (this is, in fact, obvious from Fig. 64); the positive and negative sequence components are each $\sqrt{3}$ times those of the original system. A means is thus offered of eliminating the zero sequence component, enabling the positive and negative sequence components to be measured. It would be beyond the scope of this book to consider in detail the practical applications of symmetrical component theory, but one simple example may be of interest, namely that of measuring the positive and negative sequence components of voltage in a 3-phase system. For this purpose two identical voltage transformers are required and two impedances of the same ohmic value, one being more inductive than the other to the extent that its phase angle is 60° greater. An indicating instrument of the voltmeter type (impedance Z) is also required. The circuit is successively as shown in Figs. 65A and 65B.

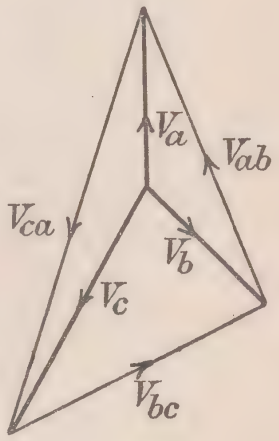


FIG. 64.

(a)
$$kV_{ab} = I_1Z_1 + IZ$$
$$kV_{ca} = I_2Z_2 + IZ$$
$$a^{\frac{1}{6}}kV_{ab} = I_1Z_2 + a^{\frac{1}{6}}IZ$$
$$k(a^{\frac{1}{6}}V_{ab} + V_{ca}) = I\{Z_2 + (1 + a^{\frac{1}{6}})Z\}$$
$$= k(\sqrt{3}a^{\frac{3}{6}}V_{a1} + \sqrt{3}a^{\frac{1}{6}}V_{a2} + \sqrt{3}a^{-\frac{7}{6}}V_{a1} + \sqrt{3}a^{\frac{7}{6}}V_{a2})$$

but $a^{\frac{1}{6}} + a^{\frac{7}{6}} = 0$ and $a^{\frac{3}{6}} + a^{-\frac{7}{6}} = \sqrt{3}a$.

Hence $3kaV_{a1} = I\{Z_2 + (1 + a^{\frac{1}{2}})Z\}$

$$V_{a1} = I\left\{\frac{Z_2 + (1 + a^{\frac{1}{2}})Z}{3ka}\right\} \quad . \quad . \quad . \quad (172)$$

but $\{Z_2 + Z(1 + a^{\frac{1}{2}})\}/3ka$ is a circuit constant and therefore the instrument reading is proportional to the positive sequence component of voltage.

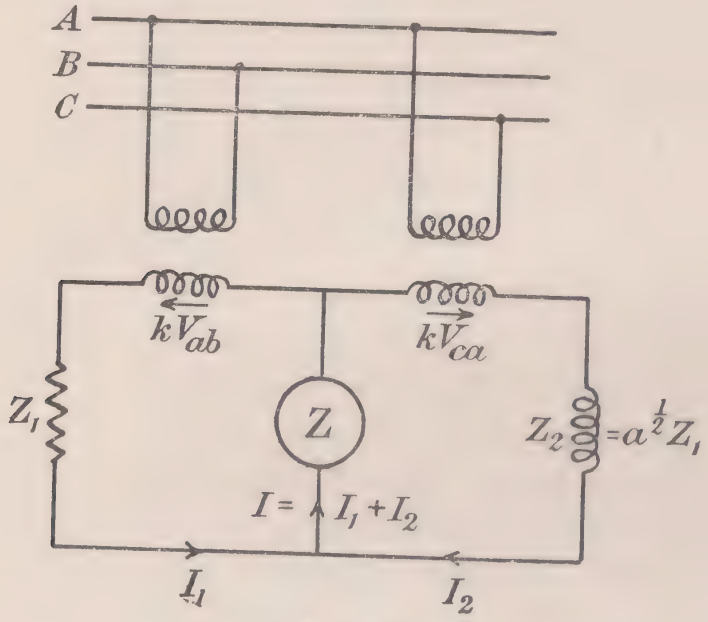


FIG. 65A.

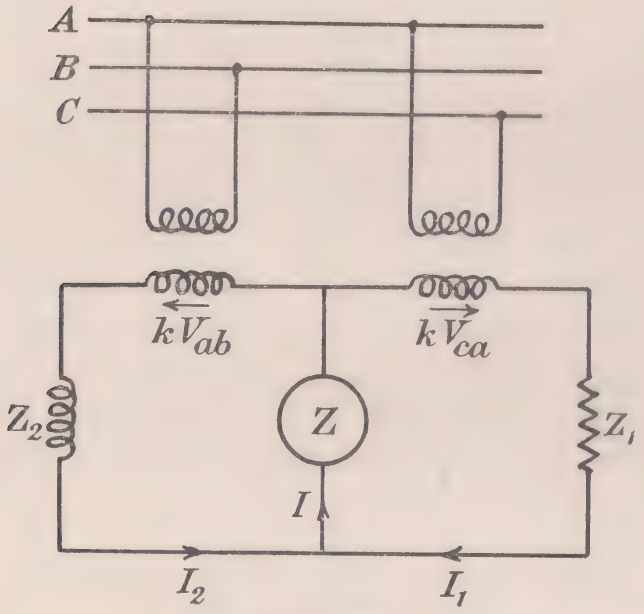


FIG. 65B.

(*b*)
$$kV_{ab} = I_2Z_2 + IZ$$
$$kV_{ca} = I_1Z_1 + IZ$$
$$a^{\frac{1}{2}}kV_{ca} = I_1Z_2 + a^{\frac{1}{2}}IZ$$
$$k(V_{ab} + a^{\frac{1}{2}}V_{ca}) = I\{Z_2 + (1 + a^{\frac{1}{2}})Z\}$$
$$= k(\sqrt{3}a^{\frac{1}{4}}V_{a1} + \sqrt{3}a^{-\frac{1}{4}}V_{a2} + \sqrt{3}a^{-\frac{5}{4}}V_{a1} + \sqrt{3}a^{\frac{9}{4}}V_{a2})$$

but $a^{\frac{1}{4}} + a^{-\frac{5}{4}} = 0$ and $a^{-\frac{1}{4}} + a^{\frac{9}{4}} = -\sqrt{3}a$.

Hence

$$V_{a2} = -I\left\{\frac{Z_2 + (1 + a^{\frac{1}{2}})Z}{3ka}\right\} \quad . \quad . \quad . \quad (173)$$

The circuit constant is the same as in the previous case and with this connection the reading of the instrument is proportional to the negative sequence component of voltage.

Power in an Unbalanced 3-phase System.

It is clear that no power can be associated with either the positive or negative sequence components of voltage and the zero sequence components of current (or vice versa) since the sum of the components in either a positive or negative sequence is zero, and all the components of the zero sequence are equal. Again, no resultant power is associated with voltages and currents of opposite sequence. For instance if

$$v_{a1} = \sqrt{2}V \cos \omega t \, ; \qquad v_{b1} = \sqrt{2}V \cos\left(\omega t - \frac{2\pi}{3}\right) ;$$

$$v_{c1} = \sqrt{2}V \cos\left(\omega t - \frac{4\pi}{3}\right) ; \qquad i_{a2} = \sqrt{2}I \cos(\omega t - \phi) ;$$

$$i_{b2} = \sqrt{2}I \cos\left(\omega t - \phi + \frac{2\pi}{3}\right) ; \qquad i_{c2} = \sqrt{2}I \cos\left(\omega t - \phi + \frac{4\pi}{3}\right) ;$$

$$\text{then } v_{a1}i_{a2} + v_{b1}i_{b2} + v_{c1}i_{c2} = VI\left\{\cos(2\omega t - \phi) + \cos\phi \right.$$

$$\left. + \cos(2\omega t - \phi) + \cos\left(\phi - \frac{4\pi}{3}\right) + \cos(2\omega t - \phi) + \cos\left(\phi - \frac{8\pi}{3}\right)\right\}$$

$$= 3VI \cos(2\omega t - \phi)$$

There is considerable oscillating power but no resultant. It therefore follows that the total *average* power can be considered as being due to the sum of the powers in the positive sequence, the negative sequence and the zero sequence.

CHAPTER VIII

PARTIAL DIFFERENTIATION

When a function depends upon two or more independent variables a change in the value of the function may be produced by a change in any of the variables. So long as the function is continuous and completely defined by the variables, the order of change of the variables does not matter. Any order which is convenient may be legitimately assumed. Thus if

$$u = f(x, y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

changes in u may be effected by changes in x or y, or both. It is convenient to consider any general change as consisting of separate and independent changes in x and y. If, while y remains unchanged, x is increased by a very small amount δx, the change in u is δx multiplied by the rate of change of u with respect to x under this condition. This particular rate of change of u with respect to x, x being the only variable changing, is called the *partial differential coefficient* of u with respect to x and is written $\frac{\partial u}{\partial x}$. Similarly the rate of change of u with respect to y when x is constant is $\frac{\partial u}{\partial y}$.

Equation 174 represents a continuous surface for which, at the point xyu and its immediate neighbourhood, the tangent plane may be substituted. In moving from x,y to $x + \delta x$, $y + \delta y$ along this plane the change in u is

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \quad . \quad . \quad . \quad . \quad . \quad (175)$$

This relation which is exact for the tangent plane is only correct to the first order of small quantities for the surface defined by equation 174. It, however, becomes exact when δx and δy are indefinitely diminished giving

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad . \quad . \quad . \quad . \quad . \quad (176)$$

du is the *total differential* of u.

Successive Partial Differentiation.

$\frac{\partial u}{\partial x}$ having been obtained, this partial derivative may again be partially differentiated with respect to x. The result is written $\frac{\partial^2 u}{\partial x^2}$. Or again $\frac{\partial u}{\partial x}$ may be partially differentiated with respect to y, written $\frac{\partial^2 u}{\partial y\,\partial x}$. Thus

if $$u = 2x^3 + 3x^2y + 4xy^2 + 5y^3$$

$$\frac{\partial u}{\partial x} = 6x^2 + 6xy + 4y^2\,; \quad \frac{\partial^2 u}{\partial x^2} = 12x + 6y$$

and $$\frac{\partial^2 u}{\partial y\,\partial x} = 6x + 8y.$$

Again $$\frac{\partial u}{\partial y} = 3x^2 + 8xy + 15y^2\,; \quad \frac{\partial^2 u}{\partial y^2} = 8x + 30y$$

and $$\frac{\partial^2 u}{\partial x\,\partial y} = 6x + 8y.$$

For this expression it is seen that

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} \quad . \quad . \quad . \quad . \quad . \quad (177)$$

This result is true generally.

A closer approximation than equation 175 for the surface represented by equation 174 can be obtained by taking account of the fact that the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ do not remain strictly constant as x changes to $x + \delta x$ and y to $y + \delta y$. Thus at $x + \delta x$, y the partial differential coefficient of u with respect to x becomes $\frac{\partial u}{\partial x} + \delta x \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} + \delta x \,.\, \frac{\partial^2 u}{\partial x^2}$. As δx is small it may be assumed that the average value of the rate of change of u with respect to x between x, y and $x + \delta x$, y is $\frac{\partial u}{\partial x} + \frac{1}{2}\delta x \frac{\partial^2 u}{\partial x^2}$. Again at $x + \delta x$, y the partial rate of change of u with respect to y is $\frac{\partial u}{\partial y} + \delta x \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial y} + \delta x \frac{\partial^2 u}{\partial x\,\partial y}$. To the same degree of approximation as before the average rate

of change of u with respect to y between $x + \delta x$, y and $x + \delta x$, $y + \delta y$ is $\frac{\partial u}{\partial y} + \delta x \frac{\partial^2 u}{\partial x\, \partial y} + \frac{1}{2}\delta y \frac{\partial^2 u}{\partial y^2}$. Thus for the *surface*

$$\delta u = \left(\frac{\partial u}{\partial x} + \tfrac{1}{2}\delta x \frac{\partial^2 u}{\partial x^2}\right)\delta x + \left(\frac{\partial u}{\partial y} + \delta x \frac{\partial^2 u}{\partial x\, \partial y} + \tfrac{1}{2}\delta y \frac{\partial^2 u}{\partial y^2}\right)\delta y \qquad (178)$$

The excess of δu for the surface over δu for the tangent plane, to the second order of small quantities is, from 175 and 178,

$$\lambda = \tfrac{1}{2}(\delta x)^2 \frac{\partial^2 u}{\partial x^2} + \delta x\, \delta y \frac{\partial^2 u}{\partial x\, \partial y} + \tfrac{1}{2}(\delta y)^2 \frac{\partial^2 u}{\partial y^2} \quad . \quad (179)$$

But if instead of proceeding from x,y to $x + \delta x$, $y + \delta y$ through $x + \delta x$, y one proceeds through x, $y + \delta y$ then the result obtained for λ is

$$\lambda = \tfrac{1}{2}(\delta y)^2 \frac{\partial^2 u}{\partial y^2} + \delta y\, \delta x \frac{\partial^2 u}{\partial y\, \partial x} + \tfrac{1}{2}(\delta x)^2 \frac{\partial^2 u}{\partial x^2} \quad . \quad . \quad (180)$$

whence

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$$

Total Differential Coefficient.

If x and y are functions of a single variable v then $du = \frac{du}{dv}dv$; $dx = \frac{dx}{dv}dv$; $dy = \frac{dy}{dv}dv$. Substituting in equation 176 and dividing by dv one obtains

$$\frac{du}{dv} = \frac{\partial u}{\partial x}\frac{dx}{dv} + \frac{\partial u}{\partial y}\frac{dy}{dv} \quad . \quad . \quad . \quad . \quad (181)$$

A particular case of importance occurs when $u = f(x, y)$ and y is a function of x; equation 181 then reduces to

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \quad . \quad . \quad . \quad . \quad . \quad (182)$$

Again, equation 182 is true whether x and y vary independently of one another or are functionally related. For instance, if $u = f(x, y) = 0$ then equation 182 gives

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad (183)$$

At first glance the sign in this equation appears incorrect. It is, however, easy to show that it is correct. In dealing with total differential coefficients, $\frac{du}{dx}$ is regarded as the limit of $\frac{\delta u}{\delta x}$.

A partial differential coefficient can only be so regarded if it is remembered that δu in $\frac{\delta u}{\delta x}$, the limit of which is $\frac{\partial u}{\partial x}$, is subject to the condition that there is no change in y. Similarly δu in $\frac{\delta u}{\delta y}$, the limit of which is $\frac{\partial u}{\partial y}$, is subject to the condition that there is no change in x. Since u does not change, it follows that δu in one case is equal but opposite to δu in the other case and so the negative sign in equation 183 is explained. In general, partial differential coefficients are considered as entities and not as ratios of differentials.

Taking a simple example, if $xy = c$, $y = c/x$ and $\frac{dy}{dx} = -\frac{c}{x^2} = -\frac{y}{x}$. This equation may be written $u = xy - c = 0$. Then $\frac{\partial u}{\partial x} = y$, $\frac{\partial u}{\partial y} = x$ whence $\frac{dy}{dx} = -y/x$.

Maxima and Minima.

When y can be expressed explicitly in terms of x the determination of maximum or minimum values of y presents no difficulty (see p. 28). If, however, y is given as an implicit function of x, such for instance as

$$x^2 + y^2 - xy + y - 1 = 0 \quad . \quad . \quad . \quad (184)$$

the determination of maximum or minimum values does necessitate y being explicitly expressed. Writing such a relation as that of equation 184 more generally as $u = f(x, y) = 0$ then

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad (185)$$

For maximum or minimum values of y, $\frac{dy}{dx} = 0$ and therefore $\frac{\partial u}{\partial x} = 0$. To distinguish between maximum and minimum values the sign of $\frac{d^2y}{dx^2}$ must be found. This can be obtained by differentiating equation 185 (using the process indicated by equation 182), giving

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial y}\frac{d^2y}{dx^2} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{dy}{dx} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = 0 \quad (186)$$

Since $\frac{dy}{dx} = 0$ this reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{d^2 y}{dx^2} = 0 \text{ or } \frac{d^2 y}{dx^2} = -\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial u}{\partial y} \quad . \quad (187)$$

For a maximum value $\frac{d^2 y}{dx^2}$ is negative and for a minimum value positive. For the particular example of equation 184

$$\frac{\partial u}{\partial x} = 2x - y\,; \quad \frac{\partial^2 u}{\partial x^2} = 2\,; \quad \frac{\partial u}{\partial y} = 2y - x + 1$$

For a maximum or minimum value $\frac{\partial u}{\partial x} = 0$ or $y = 2x$. Substituting this value gives $x^2 + 4x^2 - 2x^2 + 2x - 1 = 0$ or $3x^2 + 2x - 1 = 0$; $(3x - 1)(x + 1) = 0$

$$x = \tfrac{1}{3} \text{ or } -1$$
$$y = \tfrac{2}{3} \text{ or } -2$$
$$\frac{\partial u}{\partial y} = 2 \text{ or } -2$$
$$\frac{d^2 y}{dx^2} = -1 \text{ or } 1$$

$\frac{2}{3}$ is therefore the maximum value of y and -2 the minimum.

Three Independent Variables.

If $u = f(x, y, z)$ then equation 176 can be extended, giving

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \quad . \quad . \quad . \quad (188)$$

CHAPTER IX

FURTHER METHODS OF INTEGRATION

Integrals of the form $\int \frac{dx}{X\sqrt{Y}}$ where X and Y are linear or quadratic functions of x can be readily solved. If Y is linear the best substitution is $y = \sqrt{Y}$, whether X is linear or quadratic. If Y is quadratic and X linear the best substitution is $1/y = X$. If both X and Y are quadratic the solution is slightly more troublesome, but can be obtained by making the substitution $y = \sqrt{\frac{Y}{X}}$. Examples of these four cases will be worked out.

(i) Both X and Y Linear.

$$\int \frac{dx}{(x-1)\sqrt{x+2}}$$

Put $x + 2 = y^2$ giving $dx = 2y\,dy$ and the integral becomes

$$\int \frac{2y\,dy}{(y^2-3)y} = 2\int \frac{dy}{y^2-3} = \frac{1}{\sqrt{3}} \log \frac{y-\sqrt{3}}{y+\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \log \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$$

The same substitution serves if the integral is of the form $\int \frac{\phi(x)dx}{X\sqrt{Y}}$, X and Y being both linear and $\phi(x)$ a rational, integral, algebraic function of x. Thus in $\int \frac{(x+1)dx}{(x-1)\sqrt{x+2}}$ the same substitution as before gives

$$2\int \frac{y^2-1}{y^2-3}dy = 2\int dy + 4\int \frac{dy}{y^2-3}$$

$$= 2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$$

(ii) Y Linear, X Quadratic.

$$\int \frac{dx}{(x^2-4)\sqrt{x+1}}$$

Put $x+1=y^2$ giving $dx=2y\,dy$

$$x^2-4=y^4-2y^2-3=(y^2-3)(y^2+1)$$

and so the integral becomes

$$\int \frac{2y\,dy}{(y^2-3)(y^2+1)y}=\frac{1}{2}\left[\int\frac{dy}{y^2-3}-\int\frac{dy}{y^2+1}\right]$$

$$=\frac{1}{2\sqrt{3}}\log\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}-\frac{1}{2}\tan^{-1}\sqrt{x+1}$$

Again the same substitution may be used if the integral is of the form $\int\frac{\phi(x)dx}{X\sqrt{Y}}$ with the same limitations as before.

(iii) X Linear, Y Quadratic.

$$\int \frac{dx}{(x+1)\sqrt{x^2-2}}$$

Put $x+1=\frac{1}{y}$, giving $dx=-\frac{dy}{y^2}$

$x^2-2=\frac{1}{y^2}-\frac{2}{y}-1$ and the integral becomes

$$-\int\frac{dy}{\frac{1}{y}\cdot y^2\sqrt{\frac{1}{y^2}-\frac{2}{y}-1}}=-\int\frac{dy}{\sqrt{2-(1+y)^2}}$$

$$=-\sin^{-1}\frac{1+y}{\sqrt{2}}=-\sin^{-1}\frac{x+2}{\sqrt{2}(x+1)}$$

(iv) X and Y Both Quadratic.

$$\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}$$

Put $\frac{x^2-1}{x^2+1}=y^2$ giving $\frac{4x}{(x^2+1)^2}dx=2y\,dy$

$$dx=\frac{(x^2+1)^2}{2x}\sqrt{\frac{x^2-1}{x^2+1}}dy$$

On substituting, the integral becomes $\int \frac{\sqrt{x^2+1}}{2x} dy$. But $x^2 = \frac{1+y^2}{1-y^2}$ and the integral is therefore

$$\frac{1}{\sqrt{2}}\int \frac{dy}{\sqrt{1+y^2}} = \frac{1}{\sqrt{2}} \sinh^{-1} y = \frac{1}{\sqrt{2}} \sinh^{-1} \sqrt{\frac{x^2-1}{x^2+1}}$$

Reduction Formulæ.

Of the commonly occurring functions which are not directly integrable, some may be connected with simpler functions which are either directly integrable or are more amenable than the original functions. Thus if $x^n e^x$ be integrated by parts one obtains

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x \, dx \quad . \quad . \quad . \quad (189)$$

The troublesome factor x^n has been reduced in degree and, if n is integral, will, with continued reduction, ultimately become unity. Equation 189 is a *reduction formula.*

Reduction formulæ for $\int \cos^n \theta \, d\theta$ *and* $\int \sin^n \theta \, d\theta$.

(1) $\int \cos^n \theta \, d\theta$

If $\phi = \cos^{n-1} \theta \, . \sin \theta$

$$\frac{d\phi}{d\theta} = -(n-1)\cos^{n-2}\theta \sin^2\theta + \cos^n\theta$$

$$= -(n-1)\cos^{n-2}\theta + n\cos^n\theta$$

whence $\phi = n\int \cos^n \theta \, d\theta - (n-1)\int \cos^{n-2}\theta \, d\theta$

or

$$\int \cos^n \theta \, d\theta = \frac{1}{n}\cos^{n-1}\theta \, . \sin\theta + \frac{n-1}{n}\int \cos^{n-2}\theta \, d\theta \quad (190)$$

(ii) $\int \sin^n \theta \, d\theta$

If $\phi = \sin^{n-1} \theta \, . \cos \theta$

$$\frac{d\phi}{d\theta} = (n-1)\sin^{n-2}\theta \, . \cos^2\theta - \sin^n\theta$$

$$= (n-1)\sin^{n-2}\theta - n\sin^n\theta$$

whence $\phi = (n-1)\int \sin^{n-2}\theta\, d\theta - n\int \sin^n \theta\, d\theta$

or

$$\int \sin^n \theta\, d\theta = -\frac{1}{n}\sin^{n-1}\theta \,.\, \cos\theta + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta \quad (191)$$

The Definite Integrals $\int_0^{\frac{\pi}{2}} \cos^n \theta\, d\theta$ *and* $\int_0^{\frac{\pi}{2}} \sin^n \theta\, d\theta$, *n being Integral.*

(i) If $C_n = \int_0^{\frac{\pi}{2}} \cos^n \theta\, d\theta$ then, from equation 190, one obtains for the given limits

$$C_n = \frac{n-1}{n} C_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

or

$$C_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} C_{n-4} \quad . \quad . \quad . \quad . \quad (193)$$

If *n be even* then ultimately

$$C_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} 1\, d\theta$$

or

$$C_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad . \quad . \quad . \quad (194)$$

If *n be odd* then ultimately

$$C_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{2}{3} \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$$

or

$$C_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{2}{3} \quad . \quad . \quad . \quad . \quad . \quad (195)$$

(ii) Similarly if *n be even* one obtains from equation 191

$$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} 1\, d\theta$$

or

$$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = C_n \quad . \quad (196)$$

and if *n be odd*

$$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdot \cdot \frac{2}{3} = C_n \quad . \quad . \quad (197)$$

Investigation of $\int \sin^m\theta\cos^n\theta\,d\theta$.

If $\phi = \sin^{m-1}\theta\cos^{n+1}\theta$

$$\frac{d\phi}{d\theta} = (m-1)\sin^{m-2}\theta\cos^{n+2}\theta - (n+1)\cos^n\theta\sin^m\theta$$

$$= (m-1)\sin^{m-2}\theta\cos^n\theta(1-\sin^2\theta) - (n+1)\sin^m\theta\cos^n\theta$$

$$= (m-1)\sin^{m-2}\theta\cos^n\theta - (m+n)\sin^m\theta\cos^n\theta$$

whence $\phi = (m-1)\int\sin^{m-2}\theta\cos^n\theta\,d\theta - (m+n)\int\sin^m\theta\cos^n\theta\,d\theta$.

Hence

$$\int\sin^m\theta\cos^n\theta\,d\theta = -\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int\sin^{m-2}\theta\cos^n\theta\,d\theta \quad . \quad (198)$$

A somewhat similar expression would have been obtained if n had been reduced instead of m.

A Note upon the Gamma Function $\Gamma(p)$.

The important gamma function will be discussed a little more fully later. For present purposes it may be considered as defined by the relations

$$\Gamma(p+1) = p\Gamma(p) = p(p-1)\Gamma(p-1) \quad . \quad . \quad (199)$$

$$\Gamma(1) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

Thus $\Gamma(4) = 3\Gamma(2) = 3\,.\,2\Gamma(1) = 3\,.\,2\,.\,1 = \underline{\lfloor 3}$

Generally when p is a positive integer

$$\Gamma(p) = \underline{\lfloor (p-1)} \quad . \quad . \quad . \quad . \quad . \quad (202)$$

Again $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

The Definite Integrals $\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta$ *and* $\int_0^{\frac{\pi}{2}} \sin^n\theta$, *n being Integral.*

These integrals represented by C_n and S_n have already been investigated (p. 128). If n be even

$$C_n = S_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \sqrt{\pi}\left(\frac{n-1}{2}\cdot\frac{n-3}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\right)\Big/2\left(\frac{n}{2}\cdot\frac{n-2}{2}\cdots 2\cdot 1\right)$$

$$= \sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)\Big/2\Gamma\left(\frac{n+2}{2}\right)$$

If n be odd then

$$C_n = S_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}$$

$$= \left(\frac{n-1}{2}\cdot\frac{n-3}{2}\cdots 2\cdot 1\right)\Big/\frac{2}{\sqrt{\pi}}\left(\frac{n}{2}\cdot\frac{n-2}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right)$$

$$= \sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)\Big/2\Gamma\left(\frac{n+2}{2}\right)$$

And hence whether n is odd or even

$$C_n = S_n = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{n+2}{2}\right) \quad . \quad (203)$$

The Definite Integral $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta$.

When the integral is made definite, equation 198 gives

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2}\theta\cos^n\theta\, d\theta \quad . \quad (204)$$

On continued reduction two cases arise according as m is even or odd. If m is even then ultimately one obtains

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{1}{n+2}\int_0^{\frac{\pi}{2}} \cos^n 0\, d0$$

The coefficient of this integral may be written as

$$\left(\frac{m-1}{2}\cdot\frac{m-3}{2}\cdots\frac{1}{2}\right)\Big/\left(\frac{m+n}{2}\cdot\frac{m+n-2}{2}\cdots\frac{n+2}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{m+1}{2}\right)\Big/\left[\Gamma\left(\frac{m+n+2}{2}\right)\Big/\Gamma\left(\frac{n+2}{2}\right)\right]$$

But $\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{n+2}{2}\right)$

and hence if m is even

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

If m is odd then one obtains

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin\theta\cos^n\theta\, d\theta$$

The coefficient of this integral is

$$\Gamma\left(\frac{m+1}{2}\right)\Big/\left[\Gamma\left(\frac{m+n+2}{2}\right)\Big/\Gamma\left(\frac{n+3}{2}\right)\right]$$

and $\int_0^{\frac{\pi}{2}} \sin\theta\cos^n\theta\, d\theta = -\frac{1}{n+1}\Big[\cos^{n+1}\theta\Big]_0^{\frac{\pi}{2}} = \frac{1}{n+1}$

$$= \frac{1}{2}\cdot\frac{2}{n+1} = \frac{1}{2}\left[\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{n+3}{2}\right)\right]$$

and hence as before

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \quad . \quad (205)$$

Double Integration.

If the mass per unit area, σ, of the figure $ABCD$ (Fig. 66) is constant the total mass can be readily determined. For if $y = f(x)$ is the equation to the curve AYB then the mass of the elementary strip indicated is

$$\delta m = \sigma y\, \delta x$$

and $$m = \sigma\int_a y\, dx$$

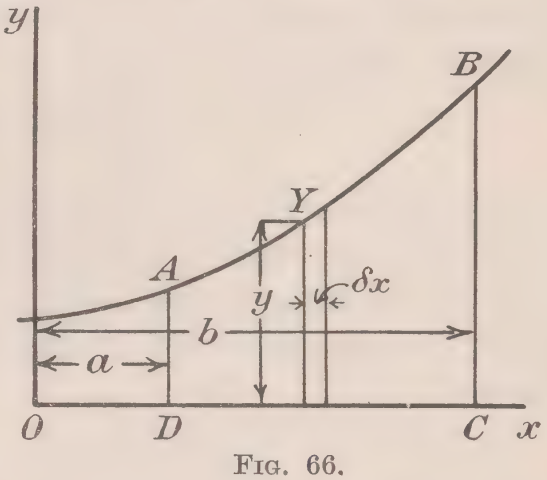


Fig. 66.

If, however, the surface density σ is variable, the determination of the mass is more complicated. The strip must be divided into elements so small that over any one the

density may be assumed to be constant. If $\sigma = \phi(x, y)$ then the mass of the element of height δy in the strip of width δx (Fig. 67) is $\phi(x, y)\delta y\ \delta x$ and the mass of the whole strip is

$$\delta x \int_{y=0}^{y=f(x)} \phi(x,\ y)dy$$

This integral having been determined as a function of x, the total mass between $x = a$ and $x = b$ is obtained by adding all such strips, giving

$$\int_a^b \left[\int_0^{f(x)} \phi(x, y)dy\right] dx$$

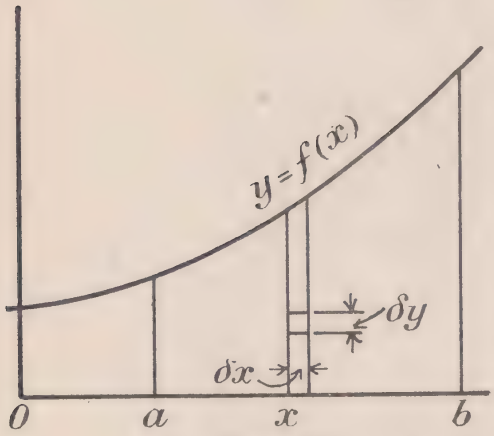


Fig. 67.

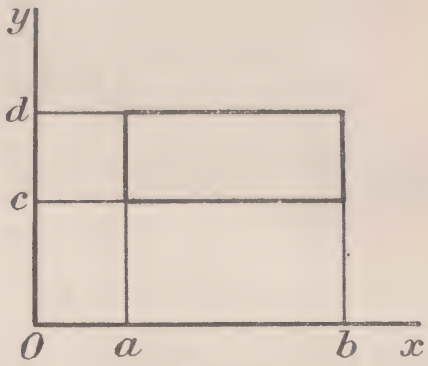


Fig. 68.

When the brackets are removed it is more usual to write the terms in a different order as

$$\int_a^b \int_0^{f(x)} \phi(x,\ y)dx\ dy$$

This order definitely indicates that the first integration is to be made with respect to y. The order is important and the point may be emphasised. Thus

$$\int_a^b \int_c^d \phi(x, y)dx\ dy$$

indicates that the first integration is with respect to y, for which the limits are c and d (these may be constants or functions of x). The second integration is with respect to x, the limits a and b being inserted after the integration has been performed. In general

$$\int_a^b \int_c^d \phi(x, y)dx\ dy \neq \int_c^d \int_a^b \phi(x, y)dy\ dx$$

The two double integrals are equal if a, b, c and d are all constants. If the area is represented by the rectangle of Fig. 68

the mass may be determined by dividing the area into strips parallel to the axis of y and the total mass is given as

$$\int_a^b \int_c^d \phi(x, y) dx\, dy$$

Or the mass may be determined by dividing the area into strips by lines parallel to the axis of x. The mass is then

$$\int_c^d \int_a^b \phi(x, y) dy\, dx$$

These two integrals must be equal.

If, however, the area is one bounded by the cubic curve $y = x^3$, the axis of x, and the ordinate $x = a$ (Fig. 69), dividing into strips parallel to the axis of y gives as the mass

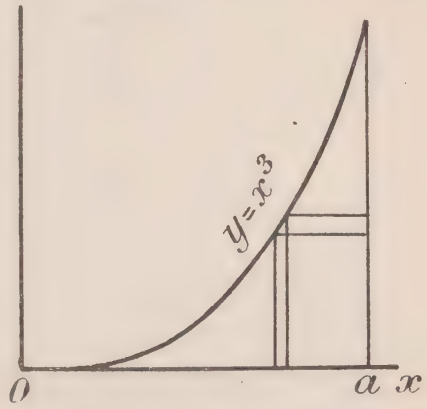


Fig. 69.

$$\int_0^a \left[\int_0^{x^3} \phi(x, y) dy \right] dx$$

or

$$\int_0^a \int_0^{x^3} \phi(x, y) dx\, dy$$

If the area be divided into strips parallel to the axis of x the mass is

$$\int_0^{a^3} \left[\int_{\sqrt[3]{y}}^a \phi(x, y) dx \right] dy$$

or

$$\int_0^{a^3} \int_{\sqrt[3]{y}}^a \phi(x, y) dy\, dx$$

Changing the order of integration has changed the limits.

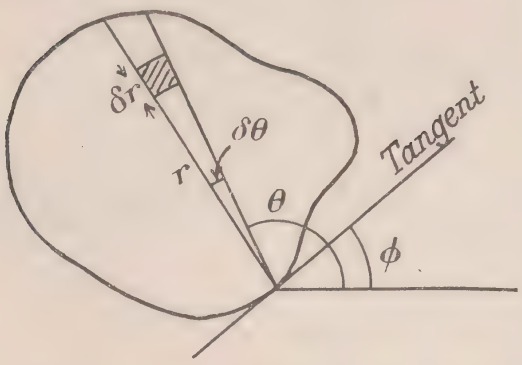


Fig. 70.

Determination of an Area from the Polar Equation.

The small element of area shown shaded in Fig. 70 is $r\,\delta\theta\,\delta r$. If the equation to the curve is $r = f(\theta)$ the area of the sector is $\delta\theta \int_0^{f(\theta)} r\, dr$. The total area includes all such

sectors between $\theta = \phi$ and $\theta = \phi + \pi$ and is equal to

$$\int_{\phi}^{\phi+\pi}\int_{0}^{f(\theta)} r\,d\theta\,dr$$

If the curve is $r = 2a\cos\theta$ (a circle of radius a having the axis as diameter) $\phi = -\frac{\pi}{2}$ and the enclosed area is

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_{0}^{2a\cos\theta} r\,d\theta\,dr = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2a^2\cos^2\theta\,d\theta$$

$$= 2a^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} = \pi a^2$$

Example.—The surface density of a circular disc, circumference $x^2 + y^2 = a^2$ is λx^2. Find the mass of the disc.

The mass is four times that of the positive quadrant. The mass of the positive quadrant is

$$\iint \lambda x^2\,dx\,dy$$

The limits for y are 0 and $\sqrt{a^2 - x^2}$; the limits for x are 0 and a. Hence the total mass is

$$4\lambda\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2\,dx\,dy = 4\lambda\int_0^a x^2\left[y\right]_0^{\sqrt{a^2-x^2}}dx$$

$$= 4\lambda\int_0^a x^2\sqrt{a^2 - x^2}\,dx.$$

Writing $x = a\sin\theta$; $dx = a\cos\theta\,d\theta$; when $x = 0$, $\theta = 0$; when $x = a$, $\theta = \frac{\pi}{2}$

and the integral becomes

$$4\lambda a^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 4\lambda a^4\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{2\Gamma(3)} = 4\lambda a^4\frac{(\frac{1}{2}\sqrt{\pi})^2}{4} = \frac{\lambda\pi a^4}{4}$$

Triple Integration.

The determination of the mass of a solid of varying density naturally leads to an integral of the form

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\phi(x, y, z)dx\,dy\,dz$$

It is evaluated in a manner similar to the double integral.

Differentiation under an Integral Sign.

Suppose a function to be integrated is $f(x, c)$ where c is independent of x, the limits of integration a and b being independent of c. Then if

$$y = \int_a^b f(x, c)dx$$

and if δy is the small change in y brought about by a small change δc in c then

$$y + \delta y = \int_a^b f(x, c + \delta c)dx$$

and

$$\frac{\delta y}{\delta c} = \int_a^b \frac{f(x, c + \delta c) - f(x, c)}{\delta c}dx$$

As δc becomes vanishingly small this becomes

$$\frac{\partial y}{\partial c} = \frac{\partial}{\partial c}\int_a^b f(x, c)dx = \int_a^b \frac{\partial}{\partial c}f(x, c)dx \quad . \quad . \quad (206)$$

Apart from its intrinsic importance this process sometimes affords a useful transformation.

Differentiation under the integral sign when one of the limits, as well as the function, is involved.

Suppose $y = \int_o^{\phi(t)} f(x, t)dx$ and that δy is the small change in y caused by a small increase δt in t. Then

$$y + \delta y = \int_o^{\phi(t+\delta t)} f(x, t + \delta t)dx = \int_o^{\phi(t)} f(x, t + \delta t)dx + \int_{\phi(t)}^{\phi(t+\delta t)} f(x, t + \delta t)dx$$

$$\frac{\partial y}{\partial t} = a + b \text{ where } a = \underset{\delta t \to o}{Lt} \int_o^{\phi(t)} \frac{f(x, t + \delta t) - f(x, t)}{\delta t}dx = \int_o^{\phi(t)} \frac{\partial}{\partial t}[f(x,t)]dx$$

$$\text{and } b = \underset{\delta t \to o}{Lt} \int_{\phi(t)}^{\phi(t+\delta t)} \frac{f(x,t + \delta t)}{\delta t}dx = \underset{\delta t \to o}{Lt} \sum_{\phi(t)}^{\phi(t+\delta t)} f(x,t + \delta t)\frac{\delta x}{\delta t}$$

Since the limits differ by an infinitesimally small quantity, the summation sign may be suppressed if δx is interpreted as the difference of the limits, and one obtains

$$b = \underset{\delta t \to o}{Lt} \frac{\phi(t + \delta t) - \phi(t)}{\delta t} f(x,t + \delta t) = \phi'(t).f(x,t)_{x=\phi(t)}$$

$$\text{Hence } \frac{\partial}{\partial t}\int_o^{\phi(t)} f(x,t)dx = \int_o^{\phi(t)} \frac{\partial}{\partial t}[f(x,t)]dx + \phi'(t).f(x,t)_{x=\phi(t)}$$

Thus if $$y = \int_o^t A(t-\tau)E(\tau)dr$$

$$\phi(t) = t,\ \phi'(t) = 1,\ f(t, r)_{r=t} = A(o)E(t)$$

and so $$\frac{\partial}{\partial t}\int_o^t A(t-\tau)E(\tau)dr = A(O)E(t) + \int_o^t A'(t-\tau)E(\tau)dr$$

(See equation 789, p. 314)

The Beta Function.

The *beta function*, sometimes called the first Eulerian integral, is defined as

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad . \quad . \quad (207)$$

Writing $1 - x = y,\ dy = -dx$

and $$B(m, n) = -\int_1^0 (1-y)^{m-1}y^{n-1}dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

or $$B(m, n) = B(n, m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

Another Form of the Beta Function.

If in equation 207 the substitution be made $x = \sin^2\theta$, then $dx = 2\sin\theta \,.\, \cos\theta\, d\theta$ and the function becomes

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\sin\theta\cos\theta\, d\theta$$

or $$B(m, n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta \quad . \quad . \quad . \quad (209)$$

or conversely

$$2\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad . \quad (210)$$

The Gamma Function.

The second Eulerian integral, or *gamma function*, has for one of its definitions

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx \quad . \quad . \quad . \quad . \quad (211)$$

Integrating by parts gives

$$\int e^{-x}x^{n-1}dx = -e^{-x}x^{n-1} + (n-1)\int e^{-x}x^{n-2}dx$$

and putting in the limits gives

$$\int_0^\infty e^{-x}x^{n-1}dx = (n-1)\int_0^\infty e^{-x}x^{n-2}dx$$

$$\therefore \Gamma(n) = (n-1)\Gamma(n-1)$$

If n is a positive integer

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)$$
$$= (n-1)(n-2) \ldots 3 \cdot 2 \cdot 1\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1$$

And so when n is a positive integer

$$\Gamma(n) = \underline{\lfloor n-1}$$

Relation between Beta and Gamma Functions.

$$\Gamma(m) = \int_0^\infty e^{-w}w^{m-1}dw$$

Putting $w = zx$ where z is independent of x gives $dw = z\,dx$ and the integral becomes

$$\Gamma(m) = \int_0^\infty e^{-zx}z^m x^{m-1}dx$$

$$\Gamma(m)e^{-z}z^{n-1} = \int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx$$

Since z and x are independent

$$\Gamma(m)\int_0^\infty e^{-z}z^{n-1}dz = \int_0^\infty\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx\,dz \quad . \quad (212)$$

To evaluate $\int_0^\infty e^{-z(1+x)}z^{m+n-1}dz$, $z(1+x)$ is put equal to y then $dz = \dfrac{dy}{1+x}$ and

$$\int_0^\infty e^{-z(1+x)}z^{m+n-1}dz = \int_0^\infty e^{-y}\,\frac{y^{m+n-1}}{(1+x)^{m+n}}\,dy = \frac{\Gamma(m+n)}{(1+x)^{m+n}}$$

whence from equation 212

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \quad . \quad . \quad (213)$$

If $x = \dfrac{1}{s} - 1$; $dx = -\dfrac{1}{s^2}ds$; $x = 0, s = \infty$; $x = \infty, s = 0$;

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = -\int_\infty^0 \frac{1}{s^2}\left(\frac{1-s}{s}\right)^{m-1}s^{m+n}ds$$

$$= \int_0^\infty s^{n-1}(1-s)^{m-1}ds = B(n,m) = B(m,n) \; . \; (214)$$

and equation 213 becomes

$$\Gamma(m) \cdot \Gamma(n) = \Gamma(m + n) \cdot B(m, n)$$

or

$$B(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m + n)} \quad . \quad . \quad . \quad . \quad (215)$$

Values of Particular Integrals in Terms of Gamma Functions.

If equation 215 be substituted in equation 210 the result immediately obtained is

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \quad . \quad (216)$$

This expression is symmetrical in m and n, as would be expected since $B(m, n) = B(n, m)$. If alternately m and n be made zero it follows that

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)} \quad (217)$$

It will be noted that this is identical with equation 203 if

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad . \quad . \quad . \quad . \quad . \quad (201)$$

a relation which had previously been assumed. The analysis above in fact constitutes a proof of equation 201. It can, however, be simply proved directly. For if n in equation 217 be made zero then the equation becomes

$$\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2})}{2\Gamma(1)} = \frac{\pi}{2}$$

But $\Gamma(1) = 1$ and therefore $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Integral Values of the Gamma Functions.

It has already been seen that if n is a positive integer $\Gamma(n) = \underline{|n - 1}$. Also $\Gamma(1) = 1$, and generally

$$\Gamma(n) = (n - 1)\Gamma(n - 1).$$

Hence it follows that $\Gamma(1) = 0 \cdot \Gamma(0)$ or $\Gamma(0) = \infty$. Similarly $\Gamma(0) = -1 \cdot \Gamma(-1)$ or $\Gamma(-1) = \infty$. Similarly $\Gamma(-2) = \infty$, etc. Generally if n is a positive integer $\Gamma(n) = \underline{|n - 1}$; if n is zero or a negative integer $\Gamma(n) = \infty$.

CHAPTER X

SIMPLE DIFFERENTIAL EQUATIONS AND EXAMPLES

Any equation involving differentials or differential coefficients is a *differential equation.* When all the differential coefficients occurring refer to a single independent variable the equation is an *ordinary* differential equation. A *partial* differential equation involves more than one independent variable and the partial differential coefficients with respect to any of them. Thus

$$CL\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = -\omega CE \sin \omega t \quad . \quad . \quad (218)$$

is an ordinary differential equation while

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad (219)$$

and

$$\frac{\partial^2 V}{\partial t^2} = c^2 \frac{\partial^2 V}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad (220)$$

are partial differential equations.

The *order* of a differential equation is the order of the highest derivative occurring in it.

The *degree* of a differential equation is the degree of the highest derivative, when the differential coefficients are free from radicals and fractions. Thus

$$CL\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = 0 \quad . \quad . \quad . \quad . \quad (221)$$

is of the second order and the first degree;

$$L\frac{di}{dt} + Ri = E \quad . \quad . \quad . \quad . \quad . \quad (222)$$

is of the first order and first degree;

$$\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad (223)$$

is of the second order and second degree.

The equations to be considered first are those involving one dependent variable and one independent variable.

A *solution* (or *integral*, or *primitive*) of a differential equation is a relation between the variables which satisfies the equation. Thus $i = \frac{E}{R}$ is a solution of equation 222 ; so is

$$i = Ae^{-\frac{R}{L}t} + \frac{E}{R}$$

where A is a constant. By giving different values to A, different solutions are obtained ; if A is made zero the particular solution $i = \frac{E}{R}$ occurs.

Again, a solution of

$$\frac{d^2i}{dt^2} + 25\frac{di}{dt} + 100i = 0 \quad . \quad . \quad . \quad . \quad (224)$$

is $i = e^{-5t}$; another solution is $i = e^{-20t}$; further solutions are $i = Ae^{-5t}$ and $i = Be^{-20t}$. A more general solution than any of the foregoing is

$$i = Ae^{-5t} + Be^{-20t}$$

This solution includes all the others as particular cases. The arbitrary constants A and B, introduced into the general solution, are *constants of integration*. Before determining the number of constants of integration which must occur in the most general form of the solution it may be useful to consider their physical significance. Thus, in Fig. 71, if there is a current i circulating, the potential difference between the terminals of the condenser being v,

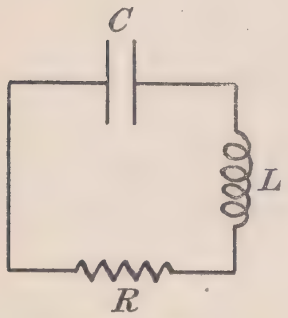


FIG. 71.

then

$$v = iR + L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad (225)$$

But the current i flowing out of the condenser is equal to the rate at which its charge is diminishing or

$$i = -\frac{dq}{dt} = -\frac{d}{dt}(Cv) = -C\frac{dv}{dt} \quad . \quad . \quad (226)$$

Differentiating equation 225 and substituting from equation 226 gives

$$CL\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = 0 \quad . \quad . \quad . \quad . \quad (221)$$

If in a particular case $L = 100$ henries, $C = 100\ \mu F$, $R = 2500$ ohms then the equation becomes

$$0{\cdot}01\frac{d^2i}{dt^2} + 0{\cdot}25\frac{di}{dt} + i = 0$$

or

$$\frac{d^2i}{dt^2} + 25\frac{di}{dt} + 100i = 0$$

which is equation 224 above. This equation expresses the most general relation which must exist in the circuit of Fig. 71 when a current flows in it. The solution given above, namely

$$i = Ae^{-5t} + Be^{-20t} \quad . \quad . \quad . \quad . \quad (227)$$

will be seen later to be a perfectly general solution. Substituting this in equation 225 gives as the potential difference across the condenser

$$v = 2000Ae^{-5t} + 500Be^{-20t} \quad . \quad . \quad (228)$$

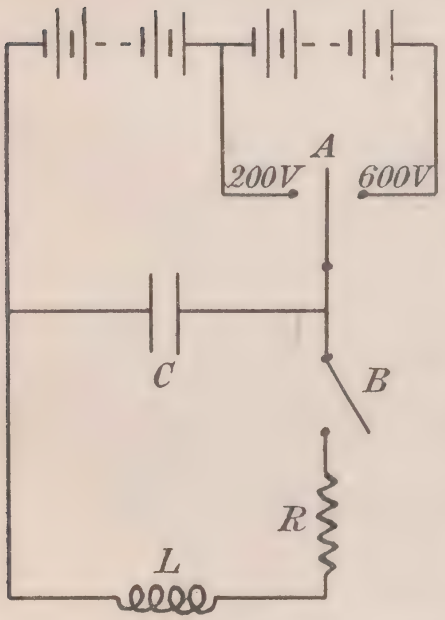


Fig. 72.

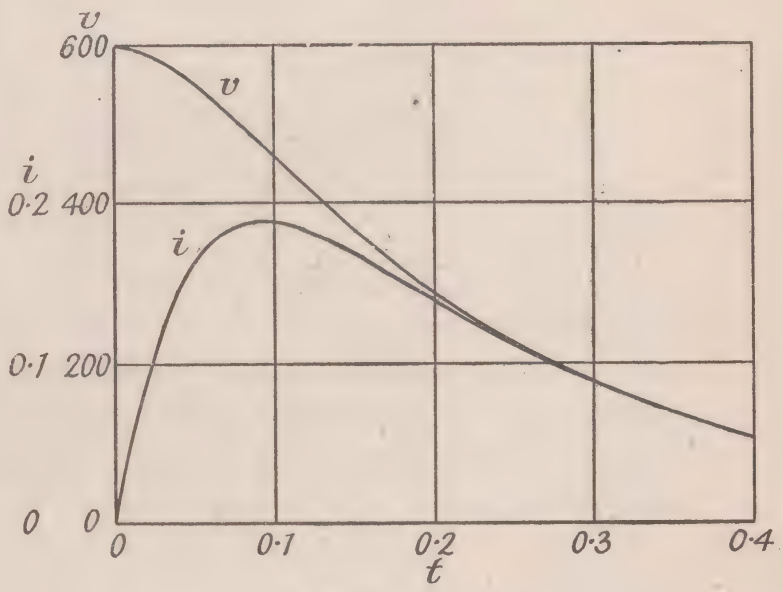


Fig. 73.

The constants A and B are determined by the actual experimental conditions. Two examples illustrate this.

(*a*) Switch A (Fig. 72) is closed on the 600-volt tapping, charging the condenser; switch B is open so that the circuit is dead.

At $t = 0$ switch A is opened and B closed. The initial conditions are, $t = 0$, $i = 0$, $v = 600$. Substituting in equations 227 and 228 gives

$$0 = A + B\,; \quad 600 = 2000A + 500B$$

or $$A = 0{\cdot}4\,; \quad B = -0{\cdot}4$$

The current in the circuit and the voltage across the condenser are given by

$$i = 0{\cdot}4(e^{-5t} - e^{-20t}) \quad . \quad . \quad . \quad . \quad (229)$$

and

$$v = 800e^{-5t} - 200e^{-20t} \quad . \quad . \quad . \quad (230)$$

These curves are plotted in Fig. 73. The current rises to $0{\cdot}1A$ in approximately 0·0218 sec., the voltage then having fallen to approximately 588.

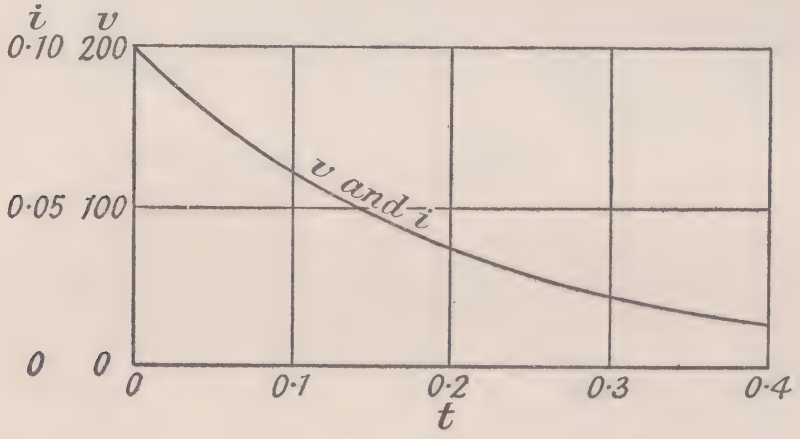


FIG. 74.

(*b*) If at the instant indicated immediately above, when the current is $0{\cdot}1A$ and the voltage is 588, the switch A is momentarily closed on the 200-volt tapping, reducing the voltage of the condenser instantaneously to 200 volts then, reckoning time from this instant, the new conditions of the circuit are,

$$t = 0, \; i = 0{\cdot}1, \; v = 200$$

Substituting in equations 227 and 228 gives

$$0{\cdot}1 = A + B\,; \quad 200 = 2000A + 500B$$

or $$A = 0{\cdot}1, \; B = 0.$$

The different values of the constants of integration should be noted. Subsequent conditions in the circuit are given by $i = 0{\cdot}1e^{-5t}$ and $v = 200e^{-5t}$ shown in Fig. 74. These are

interesting conditions. The active voltage in the circuit must be $250\varepsilon^{-5t}$; $200\varepsilon^{-5t}$ is supplied by the condenser, $50\varepsilon^{-5t}$ by the falling current in the inductance.

Number of Constants of Integration.

The most general solution of a differential equation contains as many arbitrary constants as the order of the equation. This may be seen by considering the derivation of the differential equation. To eliminate n constants, $n + 1$ equations are required. These equations are given by the solution, and n successive differentiations of the solution, whence a differential equation of the nth order is obtained.

In most differential equations the general solution (or complete primitive) includes all solutions of the equation as particular cases, the latter being derived by giving particular values to the arbitrary constants. In some exceptional cases a solution exists, called a *singular solution*, which cannot be so derived. Such solutions will not be discussed.

Equations of the First Order and of the First Degree.

Equations of the first order and of the first degree may be written as

$$P + Q\frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad (231)$$

or more symmetrically as

$$P\,dx + Q\,dy = 0 \quad . \quad . \quad . \quad . \quad . \quad (232)$$

where P and Q are functions of x and y. Such equations may belong to one or more of the following groups, (a) exact differential equations, (b) equations soluble by separation of the variables, (c) homogeneous equations, and (d) linear equations.

(a') Thus the equation

$$x\,dy + y\,dx = 0 \quad . \quad . \quad . \quad . \quad . \quad (233)$$

is an exact differential equation, for the left side is the differential of xy, and the equation may be written

$$d(xy) = 0 \text{ or } xy = c \quad . \quad . \quad . \quad . \quad (234)$$

Many equations which are not exact may be made so by multiplying by an *integrating factor*. Thus

$$L\frac{di}{dt} + Ri = V \quad . \quad . \quad . \quad . \quad . \quad (235)$$

where L, R and V are constants cannot be directly integrated as it stands, but if it be multiplied by $e^{\frac{R}{L}t}$ then it becomes

$$Le^{\frac{R}{L}t}\frac{di}{dt} + Rie^{\frac{R}{L}t} = Ve^{\frac{R}{L}t}$$

or
$$\frac{d}{dt}(Le^{\frac{R}{L}t} \cdot i) = \frac{VL}{R}\frac{d}{dt}\left(e^{\frac{R}{L}t}\right)$$

giving
$$ie^{\frac{R}{L}t} = \frac{V}{R}e^{\frac{R}{L}t} + A$$

or
$$i = \frac{V}{R} + Ae^{-\frac{R}{L}} \quad . \quad . \quad . \quad (236)$$

The factor $e^{\frac{R}{L}t}$ is the integrating factor.

Many rules may be given for the determination of integrating factors for various kinds of equations, and the determination of these factors may be interesting. From the practical point of view, however, the equations are usually more easily solved by other methods.

(b') Often equations of the first order may be written with all the terms in one variable on one side of the equation and all the terms in the other variable on the other. This separation of variables when it can be effected is one of the simplest, though perhaps not the most elegant, method of solution. Thus taking again equation 235 this may be written as

$$\frac{di}{dt} = \frac{V - Ri}{L} \text{ or } \frac{di}{i - \frac{V}{R}} = -\frac{R}{L}dt$$

Integrating gives
$$\log\left(i - \frac{V}{R}\right) = -\frac{R}{L}t + B$$

or
$$i - \frac{V}{R} = e^B e^{-\frac{R}{L}t}$$

Putting $e^B = A$ gives

$$i = \frac{V}{R} + Ae^{-\frac{R}{L}t}$$

the same result as equation 236 above.

(c') The homogeneous equation can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

The substitution is then $y = vx$.

Thus in $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ putting $y = vx$ gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1 - v^2}{2v} \text{ or } \frac{dx}{x} = dv\frac{2v}{1 - v^2}$$

or $$\log x = \log \frac{A}{1 - v^2}$$

(it is convenient when the logarithmic form is encountered to put the integration constant in the logarithmic form)

or $$x(1 - v^2) = A \text{ or } Ax = x^2 - y^2.$$

The equation $\frac{dy}{dx} = \frac{2x + y + 1}{x + 2y - 1}$ is not homogeneous as it stands but may be made so by writing

$$2x + y + 1 = 2X + Y\,; \quad x + 2y - 1 = X + 2Y$$

giving $X = x + 1\,; \quad Y = y - 1$ and the equation becomes $\frac{dY}{dX} = \frac{2X + Y}{X + 2Y}$, which is homogeneous.

(*d'*) In linear differential equations the dependent variable and its derivatives appear only in the first degree and are not multiplied together; their coefficients are all constants or functions of x. When of the first order the equation may be written in the form

$$\frac{dy}{dx} + Py = Q \quad . \quad . \quad . \quad . \quad . \quad (237)$$

where P and Q are functions of x or constants. Considering first the equation

$$\frac{dy}{dx} + Py = 0 \quad . \quad . \quad . \quad . \quad . \quad (238)$$

this may be written as $\frac{dy}{y} = -P\,dx$ giving

$$\log y = -\int P\,dx + \log A$$

$$y = Ae^{-\int P\,dx} \text{ or } ye^{\int P\,dx} = A \quad . \quad . \quad . \quad (239)$$

Differentiating equation 239 gives

$$e^{\int P\,dx}\left(\frac{dy}{dx}+Py\right)=0 \quad . \quad . \quad . \quad . \quad (240)$$

so that $e^{\int P\,dx}$ is an integrating factor of equation 238 and therefore of equation 237. Applying it to equation 237 gives

$$e^{\int P\,dx}\left(\frac{dy}{dx}+Py\right)=Qe^{\int P\,dx}$$

and the solution is

$$ye^{\int P\,dx}=\int Qe^{\int P\,dx}\,dx+B \quad . \quad . \quad . \quad (241)$$

This has already been used in solving equation 235 (p. 143).

It must be confessed that the solution of this equation is often effected without conscious use of an integrating factor, the method employed being that detailed later for the more general linear equation.

(*d''*) Certain equations which are not linear may be reduced to a linear form on substitution. One of the most important of these is the equation

$$\frac{dy}{dx}+Py=Qy^n \quad . \quad . \quad . \quad . \quad . \quad (242)$$

Dividing by y^n and multiplying by $-n+1$ this becomes

$$(-n+1)y^{-n}\frac{dy}{dx}+(-n+1)Py^{-n+1}=Q(-n+1)$$

If $v=y^{-n+1}$, $\frac{dv}{dx}=(-n+1)y^{-n}\frac{dy}{dx}$ and the equation becomes

$$\frac{dv}{dx}+(1-n)Pv=Q(1-n) \quad . \quad . \quad . \quad (243)$$

which is linear in v.

Examples.

(*a*) *Current in an Inductive Circuit.*

If an e.m.f. ε is applied to a circuit of resistance R and inductance L the current is defined by the equation

$$\varepsilon=iR+L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad (244)$$

If ε is constant or a function of t this equation is linear. Writing it in the form $\frac{di}{dt} + \frac{R}{L}i = \frac{\varepsilon}{L}$ it is seen that the integrating factor is $e^{\frac{R}{L}t}$ and the solution is

$$ie^{\frac{R}{L}t} = \int \frac{\varepsilon}{L} e^{\frac{R}{L}} \, dt + A$$

Various cases will be considered.

(i) If at $t = 0$ the circuit is connected to a constant e.m.f. E then

$$ie^{\frac{R}{L}t} = \frac{E}{L} \cdot \frac{L}{R} e^{\frac{R}{L}t} + A$$

$$i = \frac{E}{R} + Ae^{-\frac{R}{L}t}$$

At $t = 0$, $i = 0$; $\therefore A = -\frac{E}{R}$ and

$$i = \frac{E}{R}(1 - e^{-\frac{R}{L}t}) \quad . \quad . \quad . \quad . \quad (245)$$

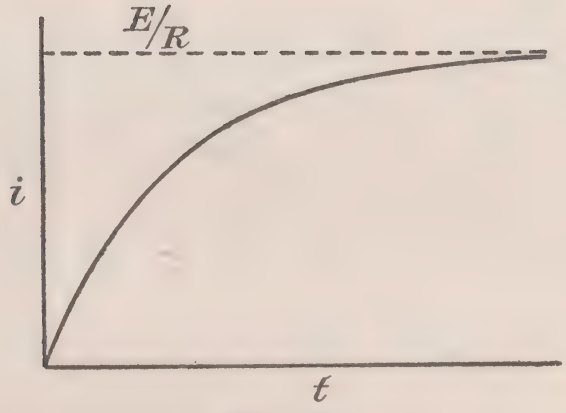


FIG. 75.

The form of this equation is shown in Fig. 75.

(ii) If at $t = 0$ when a steady current I is flowing in the circuit the source of e.m.f. is shortcircuited then

$$ie^{\frac{R}{L}t} = A$$

At $t = 0$, $i = I$ and $A = I$ giving

$$i = Ie^{-\frac{R}{L}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (246)$$

(iii) If at $t = 0$ the circuit is connected to a source of alternating e.m.f., $\varepsilon = E \cos(\omega t + \theta)$ then

$$ie^{\frac{R}{L}t} = \frac{E}{L}\int e^{\frac{R}{L}t} \cos(\omega t + \theta)dt + A$$

$$\int e^{at} \cos(\omega t + \theta)dt = \frac{1}{r}e^{at} \cos(\omega t + \theta - \alpha) \quad . \quad (247)$$

where $r = \sqrt{\omega^2 + a^2}$ and $\tan \alpha = \omega/a$ (see p. 23).

Hence

$$ie^{\frac{R}{L}t} = \frac{E}{L} \cdot \frac{L}{\sqrt{\omega^2 L^2 + R^2}} e^{\frac{R}{L}t} \cos(\omega t + \theta - \alpha) + A$$

where $\tan \alpha = \omega L/R$, and the solution is

$$i = \frac{E}{\sqrt{\omega^2 L^2 + R^2}} \cos(\omega t + \theta - \alpha) + Ae^{-\frac{R}{L}t} \quad . \quad (248)$$

at $t = 0$, $i = 0$ whence $A = -\dfrac{E}{\sqrt{\omega^2 L^2 + R^2}} \cos(\theta - \alpha)$ and

$$i = \frac{E}{\sqrt{\omega^2 L^2 + R^2}}\left[\cos(\omega t + \theta - \alpha) - \cos(\theta - \alpha)e^{-\frac{R}{L}t}\right] \quad (249)$$

The current consists of a persistent alternating component and a transient component. The latter soon dies out. In power work the former is the desired component, the latter may be a source of trouble. In acoustic work both components correctly reproduced are desirable.

Once steady conditions are established the equation becomes

$$i = \frac{E}{\sqrt{\omega^2 L^2 + R^2}} \cos(\omega t + \theta - \alpha) \quad . \quad . \quad (250)$$

$\sqrt{\omega^2 L^2 + R^2}$ is the impedance; α is the angle of lag of the current behind the voltage.

(*b*) *Current in a Capacitative Circuit.*

If an e.m.f. ε is applied to a circuit consisting of a resistance R in series with a capacitance C then

$$\varepsilon = iR + v \quad . \quad . \quad . \quad . \quad . \quad . \quad (251)$$

and

$$i = C\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

where v is the voltage on the condenser. From equation 252, equation 251 may be conveniently written

$$\varepsilon = iR + \int \frac{i}{C} dt \quad . \quad . \quad . \quad . \quad . \quad (253)$$

On differentiation this gives

$$\frac{di}{dt} + \frac{1}{CR} i = \frac{1}{R} \frac{d\varepsilon}{dt} \quad . \quad . \quad . \quad . \quad . \quad (254)$$

This is a linear equation ; the integrating factor is $e^{\frac{1}{CR}t}$ and the solution is

$$ie^{\frac{1}{CR}t} = \int \frac{1}{R} e^{\frac{1}{CR}t} \frac{d\varepsilon}{dt} dt + A \quad . \quad . \quad . \quad (255)$$

Various cases will be considered.

(i) If at $t = 0$ the circuit is connected to a constant e.m.f. E then equation 255 becomes

$$ie^{\frac{1}{CR}t} = A$$

At $t = 0$, $v = 0$ and $i = E/R$ whence $A = E/R$ giving

$$i = \frac{E}{R} e^{-\frac{1}{CR}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (256)$$

and

$$v = E(1 - e^{-\frac{1}{CR}t}) \quad . \quad . \quad . \quad . \quad . \quad (257)$$

(ii) If when the condenser is charged the battery is short-circuited then the results obtained are

$$v = Ee^{-\frac{1}{CR}t} \quad . \quad . \quad . \quad . \quad . \quad (258)$$

and

$$i = -\frac{E}{R} e^{-\frac{1}{CR}t} \quad . \quad . \quad . \quad . \quad . \quad (259)$$

(iii) If at $t = 0$ the circuit is connected to a source of alternating e.m.f., $\varepsilon = E \cos(\omega t + \theta)$ then

$$\frac{d\varepsilon}{dt} = -\omega E \sin(\omega t + \theta) = \omega E \cos\left(\omega t + \theta + \frac{\pi}{2}\right)$$

and

$$ie^{\frac{1}{CR}t} = \int \frac{\omega E}{R} e^{\frac{1}{CR}t} \left(\cos \omega t + \theta + \frac{\pi}{2}\right) dt + A$$

$$ie^{\frac{1}{CR}t} = \frac{\omega CE}{\sqrt{1 + \omega^2 C^2 R^2}} e^{\frac{1}{CR}t} \left(\cos \omega t + \theta + \frac{\pi}{2} - \alpha\right) + A$$

where $\tan \alpha = \omega CR$

or $$i = \frac{\omega CE}{\sqrt{1+\omega^2C^2R^2}}\cos\left(\omega t + \theta + \frac{\pi}{2} - \alpha\right) + Ae^{-\frac{1}{CR}t}$$

when $t = 0$, $i = 0$ and so

$$i = \frac{\omega CE}{\sqrt{1+\omega^2C^2R^2}}\left[\cos\left(\omega t + \theta + \frac{\pi}{2} - \alpha\right) - \cos\left(\theta + \frac{\pi}{2} - \alpha\right)e^{-\frac{1}{CR}t}\right] \quad . \quad . \quad (260)$$

Once steady conditions are established

$$i = \frac{\omega CE}{\sqrt{1+\omega^2C^2R^2}}\cos\left(\omega t + \theta + \frac{\pi}{2} - \alpha\right) \quad . \quad (261)$$

The impedance is $\frac{\sqrt{1+\omega^2C^2R^2}}{\omega C}$ or $\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$. The current *leads* the voltage by $\frac{\pi}{2} - \alpha$ or $\tan^{-1}\frac{1}{\omega CR}$.

Time Constant.

An interesting quantity arises out of the cases considered above, (*a*) (i) and (ii) and (*b*) (i) and (ii). In each case a quantity changes exponentially from a value M to a value N and may be written as

$$y = N - (N - M)e^{-\frac{1}{k}t} \quad . \quad . \quad . \quad . \quad (262)$$

When $t = 0$, $y = M$; when $t = \infty$, $y = N$. The constant k necessarily has the dimension of time.

$$\frac{dy}{dt} = \frac{N - M}{k}e^{-\frac{1}{k}t}$$

when $t = 0$, $\frac{dy}{dt} = \frac{N - M}{k}$. It therefore follows that if y were to continue to change at the rate at which it is changing at $t = 0$ then the final value would be attained in k seconds. Again at $t = t_1$, $\frac{dy}{dt} = \frac{N - M}{k}e^{-\frac{1}{k}t_1}$ and $y = N - (N - M)e^{-\frac{1}{k}t_1}$.

Were y to continue to change at the rate at which it is changing at $t = t_1$ then the final value N would be attained in $\left[(N - M)e^{-\frac{1}{k}t_1}\right] \div \left[\frac{N - M}{k}e^{-\frac{1}{k}t_1}\right] = k$ seconds after t_1. The estimated time to the final condition from the rate of change at any instant is therefore the constant time k. k is called the *time constant.*

CHAPTER XI

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In linear differential equations the dependent variable and its derivatives occur only in the first degree and are not multiplied together. The general form is

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} \ldots P_n y = X \quad . \quad (263)$$

where X, P_1, P_2 . . . P_n, are all constants or functions of x. The equation

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} \ldots P_n y = 0 \quad . \quad (264)$$

will first be considered.

If y_1 be a solution of equation 264, then clearly $a_1 y_1$, where a_1 is an arbitrary constant, is also a solution. Again if y_2 is a solution then $a_2 y_2$ is also a solution. Similarly if y_3, y_4 . . . y_n be solutions then substitution shows that

$$y = a_1 y_1 + a_2 y_2 + a_3 y_3 \ldots a_n y_n \quad . \quad . \quad (265)$$

is a solution and if y_1, y_2, y_3 . . . etc. are linearly independent then equation 265 is the complete solution of equation 264 since it contains n arbitrary constants.

If $y = u$ is a solution of equation 263

and $v = a_1 y_1 + a_2 y_2 \ldots a_n y_n$

then $y = u + v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (266)$

is a solution of equation 263, for the substitution of v in the first member of equation 263 gives zero and the substitution of u gives X. As the solution $y = u + v$ contains n arbitrary constants it is the complete solution of equation 263. The part v is called the *complementary function* ; u is the *particular integral*. The complete or general solution is the sum of the complementary function and the particular integral.

The Equation with Constant Coefficients ; second member zero.

The equation with constant coefficients has most important applications in oscillations of all kinds, electrical, mechanical and acoustic. Attention will first be directed to the determination of the complementary function. The equation to be solved is

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} \ldots P_n y = 0. \quad . \quad (264)$$

where $P_1, P_2 \ldots P_n$ are all constants. Substituting e^{mx} for y the first member becomes

$$(m^n + P_1 m^{n-1} + P_2 m^{n-2} \ldots P_n)e^{mx}$$

and this will be zero if

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} \ldots P_n = 0 \quad . \quad . \quad (267)$$

This is called the auxiliary equation. If its roots are $m_1, m_2 \ldots m_n$ then the solution of equation 264 is

$$y = a_1 e^{m_1 x} + a_2 e^{m_2 x} \ldots a_n e^{m_n x} \quad . \quad . \quad . \quad (268)$$

Examples.

1. To solve $\frac{d^2 i}{dt^2} + 25\frac{di}{dt} + 100i = 0$ (equation 224, p. 140).

The auxiliary equation is

$$m^2 + 25m + 100 = 0$$
$$(m + 5)(m + 20) = 0\,; \quad m = -5 \text{ or } -20$$

whence the general solution is $y = a_1 e^{-5t} + a_2 e^{-20t}$.

2. Solve $\frac{d^2 y}{dx^2} - n^2 y = 0$

The auxiliary equation is $m^2 = n^2\,; \quad m = \pm n$. The solution is therefore $y = c_1 e^{nx} + c_2 e^{-nx}$.

It is often convenient to express this in another form

$$e^{nx} = \cosh nx + \sinh nx$$
$$e^{-nx} = \cosh nx - \sinh nx$$

whence $y = (c_1 + c_2)\cosh nx + (c_1 - c_2)\sinh nx$

or $y = A \cosh nx + B \sinh nx$.

3. Solve $\frac{d^2 x}{dt^2} + n^2 x = 0$ (the equation to a simple harmonic motion).

The auxiliary equation is $m^2 = -n^2$; $m = \pm jn$. The solution is therefore $x = c_1e^{jnt} + c_2e^{-jnt}$ but

$$e^{jnt} = \cos nt + j \sin nt$$
$$e^{-jnt} = \cos nt - j \sin nt$$

whence $x = (c_1 + c_2) \cos nt + j(c_1 - c_2) \sin nt$

or $x = A \cos nt + B \sin nt.$

4. Solve $\dfrac{d^4y}{dx^4} - \gamma^4 y = 0$ (an equation occurring in the study of the forced oscillations of reeds—*Phil. Mag.*, May 1930, p. 883).

The auxiliary equation is $m^4 = \gamma^4$; $m = \pm\gamma$ or $\pm j\gamma$. The solution is therefore

$$y = X \cosh \gamma x + Y \sinh \gamma x + Z \cos \gamma x + W \sin \gamma x.$$

Equal Roots in the Auxiliary Equation.

When two roots of the auxiliary equation are equal, direct substitution in equation 268 would give a solution containing only $n - 1$ arbitrary constants since if $m_1 = m_2$ the terms $c_1e^{m_1x} + c_2e^{m_2x}$ may be written ce^{m_1x} where $c = c_1 + c_2$. The two roots may, however, be considered to differ by a quantity h which ultimately may be made indefinitely small. The corresponding two terms may then be written

$$c_1e^{m_1x} + c_2e^{(m_1+h)x} = e^{m_1x}(c_1 + c_2e^{hx})$$
$$= e^{m_1x}[c_1 + c_2(1 + hx + h^2R)] \text{ where R} \equiv \text{remainder}$$
$$= e^{m_1x}(A + Bx + BhR) \text{ where } A = c_1 + c_2$$

As h approaches zero A and B may remain finite and the two relevant terms become $e^{m_1x}(A + Bx)$. If the auxiliary solution has r equal roots the same reasoning gives as the relevant terms

$$e^{mx}(a_1 + a_2x + a_3x^2 \ldots a_rx^{r-1})$$

Complex Roots to the Auxiliary Equation.

The case of purely imaginary roots has already been treated in examples 3 and 4 above. When the auxiliary equation yields two roots of the form $\alpha + j\beta$ and $\alpha - j\beta$ the most convenient form of the solution is

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x) \quad \ldots \quad (269)$$

If complex roots are repeated the solution becomes

$$y = e^{\alpha x}[(A + A'x) \cos \beta x + (B + B'x) \sin \beta x] \quad (270)$$

The Operator D.

In much mathematical analysis the sheer manipulation may be eased by the adoption of symbols denoting particular processes. The symbol $\frac{d}{dx}$ is itself an example of this ; but for its use many of the expressions involved would be very cumbersome. The substitution of the symbols D for $\frac{d}{dx}$ and D^n for $\frac{d^n}{dx^n}$ would at first sight appear to be but a modest extension of this process of symbolic representation. But an entirely new view is presented when these operators, abstracted from the subjects upon which they act, are regarded as quantities to which the laws of algebra may be applied. This is the view accepted in operational mathematics. Operational mathematics is not new ; throughout the last century it has almost continuously been frowned upon by many of the more rigid mathematicians. But it is a powerful tool not likely to be rejected by engineers without a more solid reason.

It is not proposed to deal immediately with true operational methods (except to give some very simple examples). Symbolic representation will, however, naturally lead to their use.

Adopting the symbol D equations 264, 244 and 253 may be written as

$$(D^n + P_1D^{n-1} + P_2D^{n-2} \ldots + P_n)y = 0 \quad (271)$$

$$\left(D + \frac{R}{L}\right)i = \frac{\varepsilon}{L}$$

$$\left(R + \frac{1}{C}D^{-1}\right)i = \varepsilon \text{ ; here } D^{-1} \equiv \int(\ )dt$$

Again the equation $$\frac{dy}{dx} + y = x^2 \quad (272)$$

may be written $$(D + 1)y = x^2 \quad (273)$$

The determination of the particular integral in equations 272 or 273 may be given, without proof, as an example of operational methods. Equation 273 may be written as

$$y = \left(\frac{1}{D+1}\right)x^2 = (1 - D + D^2 - D^3 \ldots)x^2$$

or $$y = x^2 - 2x + 2 \quad (274)$$

Substitution shows that 274 is a solution of equation 272.

The Operator D (*continued*).

The symbolic coefficient of y in equation 271 is the same function of D as the left-hand side of equation 267 is of m. It follows that the roots of equation 267 can be determined by equating the symbolic coefficient of y in equation 271 to zero and solving for D as if it were an ordinary algebraic quantity instead of an operator. It is further seen that the complete solution of equation 267, which could be written as

$$(D - m_1)(D - m_2) \ . \ . \ . \ (D - m_n)y = 0 \quad . \quad (275)$$

if D were an algebraic quantity, is the sum of the solutions of

$$(D - m_1)y = 0, (D - m_2)y = 0 \ . \ . \ . \ (D - m_n)y = 0 \ . \quad (276)$$

The question immediately suggested is whether the perfectly legitimate process adopted in considering D as an algebraic quantity in order to solve the auxiliary equation 267 has any inherent validity when D is given its operational significance. For instance, the equation of the second order

$$\frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + aby = 0 \quad . \quad . \quad . \quad (277)$$

may be written as

$$[D^2 - (a + b)D + ab]y = 0 \quad . \quad . \quad . \quad (278)$$

Thinking only for the moment of the auxiliary equation it is legitimate to equate $D^2 - (a + b)D + ab$ to zero, giving $(D - a)(D - b) = 0$ and the solution of equation 277 as

$$y = c_1e^{ax} + c_2e^{bx}$$

But is it legitimate to write equations 277 or 278 as

$$(D - a)(D - b)y = 0 \quad . \quad . \quad . \quad . \quad (279)$$

when D is no longer another way of writing m but of writing $\frac{d}{dx}$? Examination shows that it is, for

$$\begin{aligned}(D - a)(D - b)y &= \left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right)y \\ &= \left(\frac{d}{dx} - a\right)\left(\frac{dy}{dx} - by\right) \\ &= \frac{d^2y}{dx^2} - b\frac{dy}{dx} - a\frac{dy}{dx} + aby \\ &= \frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + aby\end{aligned}$$

It follows that

$$(D - a)(D - b)y = (D - b)(D - a)y \quad . \quad . \quad (280)$$

and this relation is true, irrespective of the subject y upon which the compound operator $D^2 - (a + b)D + ab$ acts. So far then the process of abstraction is permissible and one may write

$$(D - a)(D - b) = (D - b)(D - a) \quad . \quad . \quad (281)$$

the subject being disregarded (or by the purist, understood).

The extension to the product of several linear factors or to the product of polynomial rational integral functions of D follows as a natural consequence.

The Polynomial in D.

The symbolic coefficient of y in equation 271 is a typical polynomial in D and the equation may be written

$$F(D)y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (282)$$

By $F(D)$ is meant a rational integral function of D of the form

$$F(D) = a_0 + a_1D + a_2D^2 \ . \ . \ . \ a_nD^n \quad . \quad . \quad (283)$$

If two simple polynomials $a_0+a_1D+a_2D^2$ and $b_0+b_1D+b_2D^2$ be considered operating successively upon a subject y then

$$\begin{aligned}(a_0 + a_1D + a_2D^2)(b_0 + b_1D + b_2D^2)y \\ = (a_0 + a_1D + a_2D^2)(b_0y + b_1Dy + b_2D^2y) \\ = a_0b_0y + a_0b_1Dy + a_0b_2D^2y + a_1b_0Dy + a_1b_1D^2y + a_1b_2D^3y \\ + a_2b_0D^2y + a_2b_1D^3y + a_2b_2D^4y \quad . \quad . \quad (284)\end{aligned}$$

It is seen that exactly the same result would be obtained if, considering D as an algebraic quantity, $a_0 + a_1D + a_2D^2$ and $b_0 + b_1D + b_2D^2$ were first multiplied together and the resulting polynomial in D were used as an operator. Furthermore, it is seen that the number of terms in the component polynomials need not be limited and finally, if $F(D)$ is defined by equation 283 and $f(D)$ by

$$f(D) = b_0 + b_1D + b_2D^2 \ . \ . \ . \ b_mD^m \quad . \quad . \quad (285)$$

then

$$F(D) \,.\, f(D) \,.\, y = [F(D) \times f(D)] \,.\, y = f(D) \,.\, F(D) \,.\, y \quad . \quad (286)$$

where . denotes operation and $\times$ algebraical multiplication. Operation is understood to commence with the operator on the right-hand side, next to the subject. In the case of the middle expression (equation 286), algebraical multiplication is

understood to precede operation. When abstracted equation 286 may be written

$$F(D) \,.\, f(D) = F(D) \times f(D) = f(D) \,.\, F(D) \quad . \quad (287)$$

Particular cases of interest arise when any function of D may be expressed as a polynomial of the form $F(D)$ on expanding by Taylor's Theorem or by algebraical manipulation. The legitimacy of such an expansion must be checked and its convergency (when applied) verified. In a practical problem it is unlikely that any difficulty will arise on this account. One very simple example will illustrate this point. An algebraic expression such as $(1 - a)^{-1}$ may be expanded as $1 + a + a^2 + a^3 \ldots$ so long as the infinite series obtained is convergent ($a < 1$). Then $(1 - a)(1 + a + a^2 + a^3 \ldots) = 1$. Similarly $(1 - D)(1 + D + D^2 + D^3 \ldots) = 1$, it being understood that operating upon the subject with $(1+D+D^2+D^3 \ldots)$ gives a result which is finite. As an example if the subject is $y = x^3$ then

$(1+D+D^2+D^3+D^4 \ldots)x^3 = x^3+3x^2+6x+6$

and

$$(1-D)(x^3+3x^2+6x+6) = x^3+3x^2+6x+6-3x^2-6x-6 = x^3$$

giving

$(1-D)(1+D+D^2+D^3 \ldots) = 1$

The Particular Integral.

Equation 263 may be written

$$F(D)y = X \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (288)$$

where $F(D)$ is the polynomial already defined. Then the particular integral is

$$y = \frac{1}{F(D)} X \quad . \quad . \quad . \quad . \quad . \quad (289)$$

If $\frac{1}{F(D)}$ can be expressed as a rational integral function of D (an example is given above) then the process of integration can be changed into one of differentiation which can always be performed. Herein lies the real strength of operational methods. In the easier cases, however, other simple methods often suffice. It is proposed at this stage to deal chiefly with these, giving some of the more general operational methods

later. The definition of $\frac{1}{F(D)}$ is implied in equations 288 and 289, for operating upon equation 289 with $F(D)$ gives

$$F(D)y = F(D) \cdot \frac{1}{F(D)} X = X \quad . \quad . \quad . \quad (290)$$

If $F(D)$ can be broken into factors

$$F(D) = f_1(D) \cdot f_2(D) \ . \ . \ . \ f_n(D) \quad . \quad . \quad . \quad (291)$$

then it is permissible to write

$$\frac{1}{F(D)} = \frac{1}{f_1(D) \cdot f_2(D) \ . \ . \ . \ f_n(D)} \quad . \quad . \quad . \quad (292)$$

for from equation 290

$$1 = f_1(D) \cdot \frac{1}{f_1(D)}$$

$$= f_2(D) \cdot f_1(D) \cdot \frac{1}{f_1(D)} \cdot \frac{1}{f_2(D)}$$

$$= f_3(D) \cdot f_2(D) \cdot f_1(D) \cdot \frac{1}{f_1(D)} \cdot \frac{1}{f_2(D)} \cdot \frac{1}{f_3(D)}$$

and finally $$F(D) \frac{1}{f_1(D) \cdot f_2(D) \ . \ . \ . \ f_n(D)} = 1$$

It follows by the same reasoning that $\frac{1}{F(D)}$ may be broken into partial fractions by ordinary algebraical methods giving

$$\frac{1}{F(D)} = \frac{N_1}{f_1(D)} + \frac{N_2}{f_2(D)} \ . \ . \ . + \frac{N_n}{f_n(D)} \quad . \quad . \quad (293)$$

The typical term in equation 293 is of the form

$$\frac{N_r}{f_r(D)} = \frac{N}{D - a}$$

$\frac{N}{D-a} X$ is the particular integral of the equation $\frac{dy}{dx} - ay = NX$, which is a linear differential equation of the first order. Its solution is, from equation 241, p. 146,

$$y = Ne^{ax} \int e^{-ax} X \, dx \quad . \quad . \quad . \quad . \quad (294)$$

In operational form this may be written

$$(D - a)^{-1} \equiv e^{ax} D^{-1} e^{-ax} \quad . \quad . \quad . \quad . \quad (294\text{A})$$

Special Methods of obtaining the Particular Integral.

For certain types of function X the particular integral may be obtained more simply than by the general method given in the preceding paragraphs. Cases which admit of simple solutions are $X = e^{ax}$; $X = e^{ax}V$ (where V is a function of x) ; $X = x^n$, n being a positive integer ; $X = \sin ax$ or $X = \cos ax$.

(i) $X = e^{ax}$

$De^{ax} = ae^{ax}$; $D^2e^{ax} = a^2e^{ax}$ and generally $D^ne^{ax} = a^ne^{ax}$. Hence $F(D)e^{ax} = F(a)e^{ax}$

$$\frac{1}{F(D)} \cdot F(D)e^{ax} = \frac{1}{F(D)} \cdot F(a)e^{ax} = F(a)\frac{1}{F(D)}e^{ax}$$

But $\dfrac{1}{F(D)} \cdot F(D)e^{ax} = e^{ax}$ and therefore

$$\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax} \quad . \quad . \quad . \quad . \quad . \quad (295)$$

unless $F(a) = 0$ when an infinite value for the particular integral is obtained.

If $F(a) = 0$, then $D - a$ must be a factor of $F(D)$. In this case $\dfrac{1}{F(D)}$ may be written $\dfrac{1}{D-a} \cdot \dfrac{1}{\phi(D)}$.

Then $\dfrac{1}{F(D)}e^{ax} = \dfrac{1}{D-a} \cdot \dfrac{1}{\phi(D)}e^{ax} = \dfrac{1}{D-a} \cdot \dfrac{e^{ax}}{\phi(a)}$.

Then from equation 294,

$$\frac{1}{F(D)}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int dx = \frac{xe^{ax}}{\phi(a)} \quad . \quad . \quad (296)$$

Should $D - a$ occur twice in the factorisation of $F(D)$ then $\dfrac{1}{F(D)}$ may be written $\dfrac{1}{(D-a)^2} \cdot \dfrac{1}{\psi(D)}$ and

$$\frac{1}{F(D)}e^{ax} = \frac{1}{(D-a)^2} \cdot \frac{1}{\psi(D)}e^{ax} = \frac{1}{(D-a)^2} \cdot \frac{e^{ax}}{\psi(a)} = \frac{1}{D-a} \cdot \frac{xe^{ax}}{\psi(a)}$$

Then from equation 294

$$\frac{1}{F(D)}e^{ax} = \frac{x^2e^{ax}}{2\psi(a)} \quad . \quad . \quad . \quad . \quad . \quad (297)$$

Examples.

1. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{4x}$.

$$(D - 3)(D - 1)y = e^{4x}$$

The complementary function is $y = a_1e^{3x} + a_2e^{x}$.
The particular integral is

$$y = \frac{e^{4x}}{D^2 - 4D + 3} = \frac{e^{4x}}{16 - 16 + 3} = \frac{1}{3}e^{4x}$$

The complete solution is $y = a_1e^{3x} + a_2e^{x} + \frac{1}{3}e^{4x}$.

2. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$.

$$(D - 3)(D - 2)y = e^{3x}$$

The complementary function is $y = a_1e^{3x} + a_2e^{2x}$.
The particular integral is

$$y = \frac{e^{3x}}{(D - 3)(D - 2)} = \frac{1}{D - 3} \cdot e^{3x} = xe^{3x}$$

and the complete solution is $y = a_1e^{3x} + a_2e^{2x} + xe^{3x}$.

(ii) $X = e^{ax}V$.

If W be a function of x then

$$De^{ax}W = e^{ax}DW + ae^{ax}W = e^{ax}(D + a)W$$
$$D^2e^{ax}W = ae^{ax}(D + a)W + e^{ax}D(D + a)W = e^{ax}(D + a)^2W$$

and generally

$$D^ne^{ax}W = e^{ax}(D + a)^nW \text{ and}$$
$$F(D)e^{ax}W = e^{ax}F(D + a)W \quad . \quad . \quad . \quad (298)'$$

If $F(D + a)W = V$ then $W = \frac{1}{F(D + a)}V$ where V is also a function of x.

Substituting this value in equation 298 gives

$$F(D)e^{ax}\frac{1}{F(D + a)}V = e^{ax}V$$

or

$$\frac{1}{F(D)}e^{ax}V = e^{ax}\frac{1}{F(D + a)}V \quad . \quad . \quad (299)$$

Equation 298 is Heaviside's shifting theorem. A more general form is given in the next chapter.

(iii) $X = x^n$.

$$y = [F(D)]^{-1}x^n$$

$[F(D)]^{-1}$ is expressed as a polynomial in rising powers of D as far as D^n.

Examples.

1. Find a particular integral of

$$(D^3 + 8)y = x^4 + 2x + 1$$

$$y = \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1} . (x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 - \frac{D^3}{8} + \frac{D^6}{64} \ldots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{8}(x^4 + 2x + 1 - 3x) = \frac{1}{8}(x^4 - x - 1)$$

2. Find a particular integral of

$$(D^2 - 4D + 4)y = x^2$$

$$y = \frac{1}{4}\left(1 - D + \frac{D^2}{4}\right)^{-1} . x^2$$

$$= \frac{1}{4}\left(1 + D - \frac{D^2}{4} + D^2 \ldots\right)x^2$$

$$= \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

(iv) $X = \sin ax$ or $\cos ax$.

$$y = \frac{1}{F(D)} \begin{matrix}\sin ax \\ \cos ax\end{matrix}$$

$$D^2 \begin{matrix}\sin ax \\ \cos ax\end{matrix} = -a^2 \begin{matrix}\sin ax \\ \cos ax\end{matrix}$$

$$[D^2]^n \begin{matrix}\sin ax \\ \cos ax\end{matrix} = (-a^2)^n \begin{matrix}\sin ax \\ \cos ax\end{matrix}$$

$$f(D^2) \begin{matrix}\sin ax \\ \cos ax\end{matrix} = f(-a^2) \begin{matrix}\sin ax \\ \cos ax\end{matrix}$$

and

$$\frac{1}{f(D^2)} \begin{matrix}\sin ax \\ \cos ax\end{matrix} = \frac{1}{f(-a^2)} \begin{matrix}\sin ax \\ \cos ax\end{matrix}$$

(v) Similarly

$$\frac{1}{f(D^2)} \begin{matrix}\sinh ax \\ \cosh ax\end{matrix} = \frac{1}{f(a^2)} \begin{matrix}\sinh ax \\ \cosh ax\end{matrix}$$

Examples.

1. Find a particular integral of

$$(D^3 + D^2 - D - 1)y = \cos 2x$$

$$y = \frac{1}{D+1} \cdot \frac{1}{D^2-1} \cos 2x = \frac{D-1}{(D^2-1)^2} \cos 2x = \frac{1}{25}(D-1) \cos 2x$$

$$= -\frac{1}{25} (\cos 2x + 2 \sin 2x)$$

It is allowable to substitute -4 for D^2 at any stage thus

$$y = \frac{1}{D^3 + D^2 - D - 1} \cos 2x = \frac{1}{-4D - 4 - D - 1} \cos 2x$$

$$= -\frac{1}{5} \frac{1}{D+1} \cos 2x = \frac{D-1}{25} \cos 2x \text{ as before.}$$

2. Find a particular integral of

$$(D^2 + a^2)y = \sin ax$$

Here the method given above fails, for if D^2 is put equal to $-a^2$ an infinite value is obtained for the particular integral. To solve this $\sin ax$ is expressed in the exponential form $\frac{e^{jax} - e^{-jax}}{2j}$ and $D^2 + a^2$ is factorised as $(D + ja)(D - ja)$.

Then $$y = \frac{1}{(D+ja)(D-ja)} \frac{e^{jax} - e^{-jax}}{2j}$$

Here again the simple rule of equation 295 cannot be applied. It fails with $D + ja$ for e^{-jax} and with $D - ja$ for e^{jax}. Both rules 295 and 296 have to be used.

Writing it as

$$y = \frac{1}{2j}\left[\frac{1}{D-ja} \cdot \frac{1}{D+ja} e^{jax} - \frac{1}{D+ja} \cdot \frac{1}{D-ja} e^{-jax}\right]$$

one obtains

$$y = \frac{1}{2j}\left[\frac{1}{D-ja} \cdot \frac{1}{2ja} e^{jax} + \frac{1}{D+ja} \cdot \frac{1}{2ja} e^{-jax}\right]$$

(from equation 295)

or $$y = -\frac{1}{4a}\left[\frac{1}{D-ja} e^{jax} + \frac{1}{D+ja} e^{-jax}\right]$$

$$= -\frac{1}{4a}[xe^{jax} + xe^{-jax}] = -\frac{x}{2a} \cos ax \quad . \quad . \quad (300)$$

(from equation 296)

3. Solve $\frac{d^4y}{dx^4} - y = e^x \cos x.$

The complementary function is
$y = a_1 \cosh x + a_2 \sinh x + a_3 \cos x + a_4 \sin x$ (see example 4, p. 153).

The particular integral is given by

$$y = \frac{1}{D^4 - 1} e^x \cos x = e^x \frac{1}{(D+1)^4 - 1} \cos x$$

(from equation 299)

$$= e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x = e^x \frac{1}{1 - 4D - 6 + 4D} \cos x$$

$$= -\tfrac{1}{5} e^x \cos x.$$

The complete solution is

$$y = a_1 \cosh x + a_2 \sinh x + a_3 \cos x + a_4 \sin x - \tfrac{1}{5} e^x \cos x.$$

CHAPTER XII

MORE GENERAL OPERATIONAL METHODS

The method so far adopted in solving a differential equation of the form

$$F(D)y = X \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)$$

has been first to determine the complementary function, defined by the equation

$$F(D)y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (302)$$

and then, using elementary operational methods, to determine the particular integral from the equation

$$y = \frac{1}{F(D)}X \quad . \quad . \quad . \quad . \quad . \quad (303)$$

The necessary number of constants of integration occur in solving equation 302 and no constant of integration occurs in the solution of equation 303. The sum of the solutions of equations 302 and 303 is the general solution. For certain particular forms of X methods of solving equation 303 have been given. Operational methods, however, enable one to deal with practical functions of much greater complexity, and it is proposed to indicate briefly some of the properties of operators and their mode of application. In the space available it is not possible to deal with the methods in any detail but the amount which can be given is of considerable assistance in many practical problems and forms a convenient introduction to a more detailed study should it be desired.

Definitions and Operators.

An *operator* is a symbol used to indicate a definite mathematical process to be performed upon some function; this function upon which the operation is performed is called the *subject*. Thus the operator D acting upon $f(x)$ is meant to indicate that x is first increased by a small amount h and the new value of the function determined. From this new value

of the function the original value of the function is subtracted and the difference is then divided by h. The value of the quotient when h is made indefinitely small is the result of the operation, or

$$D \,.\, f(x) \equiv \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} \quad . \quad . \quad (304)$$

Other operators besides D and functions of D are not uncommonly employed. For instance, the operation of increasing x in $f(x)$ by h is sometimes indicated by ψ, or

$$\psi \,.\, f(x) = f(x+h) \,. \quad . \quad . \quad . \quad . \quad . \quad (305)$$

Again the process of increasing x to $x + h$ in $f(x)$ and from the new value so obtained subtracting the original value of the function is sometimes indicated by Δ, or

$$\Delta \,.\, f(x) = f(x+h) - f(x) \quad . \quad . \quad . \quad (306)$$

whence it appears that

$$\Delta \,.\, f(x) = (\psi - 1) \,.\, f(x) \quad . \quad . \quad . \quad . \quad (307)$$

Important operators in many problems are $d_1 \equiv \frac{\partial}{\partial x}$, $d_2 \equiv \frac{\partial}{\partial y}$, $d_3 \equiv \frac{\partial}{\partial z}$ and $\nabla \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \equiv d_1 + d_2 + d_3$.

Of the polynomial operators the most important is $F(D)$ already defined, and used, and to which more extended reference will be made later.

A compound operator consists of a series of operators. The subject of each is the result of all the preceding operations. The operations are performed successively, commencing with that on the right, next to the subject. For instance the successive operations of c (constant multiplier), ψ, and Δ upon $f(x)$ is indicated by $\Delta \,.\, \psi \,.\, c \,.\, f(x)$. If performed, these operations give successively

$$\begin{aligned} \Delta \,.\, \psi \,.\, c \,.\, f(x) &= \Delta \,.\, \psi \,.\, [cf(x)] = \Delta \,.\, [cf(x+h)] \\ &= cf(x+2h) - cf(x+h) \end{aligned}$$

If the operations are performed in the inverse order, the result obtained is

$$\begin{aligned} c \,.\, \psi \,.\, \Delta \,.\, f(x) &= c \,.\, \psi \,.\, [f(x+h) - f(x)] \\ &= c \,.\, [f(x+2h) - f(x+h)] = cf(x+2h) - cf(x+h) \end{aligned}$$

It is seen that in this case the order of the operators is a matter

of indifference. They are then said to be commutative. They are not always so. For instance

$$D \,.\, x \,.\, \sin x = D \,.\, x \sin x = \sin x + x \cos x$$

but $x \,.\, D \,.\, \sin x = x \,.\, \cos x = x \cos x$.

Generally the order of operation must be maintained unless it is certain that the operators are commutative.

An *inverse operator* is expressed by a negative exponent and is sufficiently defined by the relation

$$P^{-1} \,.\, P \,.\, S \equiv S \quad . \quad . \quad . \quad . \quad . \quad (308)$$

The idea of *abstraction* is an essential feature of operational mathematics. When relations exist between operators independently of the nature of the subject the operators themselves may be abstracted and treated without reference to the subject. Thus equation 307 may be written

$$\Delta \equiv \psi - 1 \quad . \quad . \quad . \quad . \quad . \quad (309)$$

and equation 308 as $P^{-1} \,.\, P \equiv 1 \quad . \quad . \quad . \quad . \quad . \quad (310)$

Little, if anything, would be gained thereby if relations between the operators were not subject to algebraical transformation. Fortunately, as has already been seen, much algebraical transformation is permissible (see p. 155). Under these circumstances the operators are said to be *interpretable*.

The Operator D and Rules relating to it.

$F(D)$ has already been defined (equation 283) as

$$F(D) \equiv a_0 + a_1 D + a_2 D^2 \ . \ . \ . \ a_n D^n \equiv \sum_{k=0}^{n} a_k D^k$$

It may be considered to include any operator in D which can be expanded by algebraical operations or otherwise to give such a form. Examples are

$$e^D = 1 + D + \frac{D^2}{\lfloor 2} + \frac{D^3}{\lfloor 3} + \ . \ . \ .$$

$$\sin D = D - \frac{D^3}{\lfloor 3} + \frac{D^5}{\lfloor 5} - \ . \ . \ .$$

$$\frac{1}{1 - D} = (1 - D)^{-1} = 1 + D + D^2 + D^3 \ . \ . \ .$$

Clearly

$$F(D) \,.\, [u + v + w \ . \ . \ .] = F(D) \,.\, u + F(D) \,.\, v + F(D) \,.\, w \quad (311)$$

Again, if $D \equiv \frac{\partial}{\partial x}$ and $p = \frac{\partial}{\partial t}$ (p is largely used in electrical theory) then

$$D + p = p + D \quad . \quad . \quad . \quad . \quad . \quad (312)$$

for $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}$$

and

$$D \,.\, p \equiv p \,.\, D \quad . \quad . \quad . \quad . \quad . \quad (313)$$

for $$\frac{\partial^2 u}{\partial x \,.\, \partial t} = \frac{\partial^2 u}{\partial t \,.\, \partial x}$$

Again it has been shown, equation 287, that

$$F(D) \,.\, f(D) = f(D) \,.\, F(D)$$

and generally operators in D are commutative.

Expansion of Inverse Operators.

Examples of the expansion of the inverse operator have already been given (p. 157 and Examples 1 and 2, p. 161). More than one method of expanding the inverse operator is, however, possible, giving results which may appear entirely different. The difficulty is, however, soon resolved. As a particular case the equation

$$\frac{dy}{dx} - ay = x^2 \quad . \quad . \quad . \quad . \quad . \quad (314)$$

will be considered. The complementary function is

$$y = Ae^{ax} \quad . \quad . \quad . \quad . \quad . \quad . \quad (315)$$

The particular integral is given by

$$y = \frac{1}{D - a} x^2 \quad . \quad . \quad . \quad . \quad . \quad (316)$$

Now $(D - a)^{-1}$ may be alternatively expanded as

$$(D - a)^{-1} = D^{-1}\left(1 - \frac{a}{D}\right)^{-1} = D^{-1}\left(1 + \frac{a}{D} + \frac{a^2}{D^2} + \ldots\right)$$

$$= \sum_{k=1}^{\infty} a^{k-1} D^{-k} \quad . \quad . \quad (317)$$

or

$$(D-a)^{-1} = -a^{-1}\left(1 - \frac{D}{a}\right)^{-1} = -a^{-1}\left(1 + \frac{D}{a} + \frac{D^2}{a^2} + \ldots\right)$$

$$= -\sum_{k=0}^{\infty} a^{-k-1} D^k \quad . \quad . \quad (318)$$

Equation 317 applied to x^2 gives an infinite series as the particular integral while equation 318 gives only three terms. Thus applying equation 317 one obtains as the complete solution

$$y = A_1 e^{ax} + \frac{x^3}{3} + \frac{ax^4}{3 \cdot 4} + \frac{a^2 x^5}{3 \cdot 4 \cdot 5} + \ldots \quad (319)$$

and applying (318) the complete solution is given as

$$y = A_2 e^{ax} - \frac{x^2}{a} - \frac{2x}{a^2} - \frac{2}{a^3} \quad \ldots \quad (320)$$

Obviously 320 is the more convenient result to use, but the question arises whether they can both be correct. Actually they can be identical if proper values are assigned to A_1 and A_2. A_1 and A_2 are determined by the given conditions of the problem defined by equation 314, and however the result is obtained a correct solution must be given by a proper choice of the constant of integration. Thus equation 319 may be written as

$$y = A_1 e^{ax} - \frac{x^2}{a} - \frac{2x}{a^2} - \frac{2}{a^3} + \left[\frac{2}{a^3} + \frac{2x}{a^2} + \frac{x^2}{a} + \frac{x^3}{3} + \frac{ax^4}{3.4} + \frac{a^2x^5}{3.4.5} + \ldots\right]$$

$$= A_1 e^{ax} - \frac{x^2}{a} - \frac{2x}{a^2} - \frac{2}{a^3} + \frac{2}{a^3} \cdot e^{ax}$$

or

$$y = \left(A_1 + \frac{2}{a^3}\right) e^{ax} - \frac{x^2}{a} - \frac{2x}{a^2} - \frac{2}{a^3} \quad \ldots \quad (321)$$

Equations 320 and 321 are identical if $A_2 = A_1 + \frac{2}{a^3}$.

Generally in the determination of the particular integral the inverse operator will be of the form $\frac{1}{F(D)}$ which may be factorised as $\frac{1}{D-a} \cdot \frac{1}{D-b} \ldots$. In some cases (examples are 2, p. 160, and 2, p. 162) it may be convenient to operate successively with these operators. More often the most convenient method is to put the expression into partial fractions. In some cases, particularly when terms are repeated, special methods may have to be employed. A case of this occurred when operating upon e^{ax} and the factor $\frac{1}{D-a}$ was

present, either singly or repeated (equations 296 and 297). Operational methods afford a convenient method of dealing with such cases. These will be considered in conjunction with certain fundamental theorems.

$F(D) \cdot e^{\phi(x)}$—Shifting.

It is shown below that

$$F(D) \cdot e^{\phi(x)} \equiv e^{\phi(x)} \cdot F[D + \phi'(x)] \quad . \quad . \quad (322)$$

It should be noted that the whole expression on either side of this equation is an operator. To clarify the situation it may be written as

$$F(D) \cdot e^{\phi(x)} \cdot S \equiv e^{\phi(x)} \cdot F[D + \phi'(x)] \cdot S \; . \quad . \quad (322\text{A})$$

where the subject S is any function of x. This theorem, propounded by Robert Murphy in 1837, in a simplified particular case reduces to Heaviside's shifting theorem.

$$\begin{aligned} D \cdot e^{\phi(x)} \cdot S &= e^{\phi(x)} \cdot D \cdot S + S \cdot \phi'(x) \cdot e^{\phi(x)} \\ &= e^{\phi(x)}[D + \phi'(x)] \cdot S \end{aligned}$$

or by abstraction

$$D \cdot e^{\phi(x)} = e^{\phi(x)} \cdot [D + \phi'(x)] \quad . \quad . \quad . \quad (323)$$

Operating on both sides with $\varepsilon^{-\phi(x)}$ gives

$$e^{-\phi(x)} \cdot D \cdot e^{\phi(x)} = D + \phi'(x) \quad . \quad . \quad . \quad (324)$$

Now operating on both sides with the identical forms gives

$$\varepsilon^{-\phi(x)} \cdot D \cdot e^{\phi(x)} \cdot e^{-\phi(x)} \cdot D \cdot e^{\phi(x)} = [D + \phi'(x)] \cdot [D + \phi'(x)]$$

Now the two centre terms of the left-hand member are reciprocal operators and so one obtains

$$e^{-\phi(x)} \cdot D^2 \cdot e^{\phi(x)} = [D + \phi'(x)]^2 \quad . \quad . \quad . \quad (325)$$

Proceeding similarly one obtains

$$e^{-\phi(x)} \cdot D^n \cdot e^{\phi(x)} = [D + \phi'(x)]^n \quad . \quad . \quad (326)$$

Giving n all integral values and using constant multipliers one can build up a rational integral function giving

$$e^{-\phi(x)} \cdot F(D) \cdot e^{\phi(x)} = F[D + \phi'(x)]$$

and

$$F(D) \cdot e^{\phi(x)} = e^{\phi x} \cdot F[D + \phi'(x)] \quad . \quad (322)$$

Adding two more operators, $F^{-1}[D + \phi'(x)]$ immediately

before the subject and $F^{-1}(D)$ as the final operator one obtains

$$F^{-1}(D) \cdot F(D) \cdot e^{\phi(x)} \cdot F^{-1}[D + \phi'(x)] = F^{-1}(D) \cdot e^{\phi(x)} \cdot F[D + \phi'(x)] \cdot F^{-1}[D + \phi'(x)]$$

or $$e^{\phi(x)} \cdot F^{-1}[D + \phi'(x)] = F^{-1}(D) \cdot e^{\phi(x)}$$

and transposing

$$F^{-1}(D) \cdot e^{\phi(x)} = e^{\phi(x)} \cdot F^{-1}[D + \phi'(x)] \quad . \quad . \quad (327)$$

Two very simple cases illustrating these operators have already been treated, namely e^{ax} and $e^{ax}V$ (pp. 159 and 160), but it is seen that the operational equation 322 and its inverse equation 327 are much more far reaching. Some particular cases will be considered. If $\phi(x) = ax$ then $\phi'(x) = a$ and the equations degenerate to

$$F(D) \cdot e^{ax} = e^{ax}F(D + a) \quad . \quad . \quad . \quad (328)$$

and

$$F^{-1}(D) \cdot e^{ax} = e^{ax}F^{-1}(D + a) \quad . \quad . \quad . \quad (329)$$

These are important operational forms. Equation 328 is Heaviside's shifting theorem.

If the subject S is unity then equation 328 further degenerates to

$$F(D) \cdot e^{ax} \cdot 1 = e^{ax} \cdot F(D + a) \cdot 1 = e^{ax} \cdot F(a)$$

and equation 329 gives

$$\frac{1}{F(D)} \cdot e^{ax} \cdot 1 = e^{ax} \cdot F^{-1}(D + a) \cdot 1 = e^{ax} \cdot F^{-1}(D + a) \cdot e^{0x} = e^{ax} \cdot F^{-1}(a)$$

which is identical with equation 295. An important identity is here disclosed namely

$$F^{-1}(D + a) \cdot 1 = F^{-1}(a) \quad . \quad . \quad . \quad . \quad (330)$$

Equations 328 and 329 fail if $D - a$ is a factor of $F(D)$. In that case if $F(D) = \phi(D) \cdot (D - a)^r$ and the subject is unity

$$\frac{1}{F(D)} \cdot e^{ax} \cdot 1 = \frac{1}{(D - a)^r} \cdot \frac{1}{\phi(D)} \cdot e^{ax} \cdot 1 = e^{ax} \cdot \frac{1}{D^r} \cdot \frac{1}{\phi(D + a)} \cdot 1$$

$$= e^{ax} \cdot \frac{1}{D^r} \cdot \frac{1}{\phi(a)} \cdot 1 = \frac{1}{\phi(a)} \cdot e^{ax} \cdot \frac{1}{D^r} \cdot 1$$

$\frac{1}{D^r} \cdot 1 = \frac{x^r}{\lfloor r}$ (obtained by integrating unity r times) and so

$$\frac{1}{F(D)} \cdot e^{ax} \cdot 1 = \frac{1}{\phi(a)} \cdot \frac{x^r e^{ax}}{\lfloor r} \quad . \quad . \quad . \quad (331)$$

This is an extension of equations 296 and 297.

The Circular and Hyperbolic Functions.

It has already been seen that

$$F(D^2) \begin{matrix} \sin ax \\ \cos ax \end{matrix} = F(-a^2) \begin{matrix} \sin ax \\ \cos ax \end{matrix} \Bigg\} \quad . \quad . \quad (332)$$

and

$$F(D^2) \begin{matrix} \sinh ax \\ \cosh ax \end{matrix} = F(a^2) \begin{matrix} \sinh ax \\ \cosh ax \end{matrix} \Bigg\} \quad . \quad . \quad . \quad (333)$$

One example involving 332 may be reworked making use of the shifting theorem. In working example 2, p. 162, difficulty was encountered because direct application of equation 332 gives an infinite value for the particular integral in the equation

$$(D^2 + a^2)y = \sin ax$$

The particular integral of this equation and of

$$(D^2 + a^2)y = \cos ax$$

can both be determined readily using operational methods, thus

$$\frac{1}{D^2 + a^2} e^{jax} = e^{jax} \frac{1}{(D + ja)^2 + a^2} \cdot 1$$

$$= e^{jax} \frac{1}{D^2 + 2jaD} \cdot 1 = e^{jax} \frac{1}{D} \cdot \frac{1}{D + 2ja} \cdot e^{0x}$$

$$= e^{jax} \cdot \frac{1}{D} \cdot \frac{1}{2ja} = e^{jax} \cdot \frac{x}{2ja}$$

That is $\frac{1}{D^2 + a^2} \cdot (\cos ax + j \sin ax)$

$$= \frac{x}{2a}(-j \cos ax + \sin ax)$$

Equating real and imaginary parts gives

$$\frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax \quad . \quad . \quad . \quad (300)$$

the result obtained previously, and

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax \quad . \quad . \quad . \quad (334)$$

The Linear Differential Equation of the First Order.

It has been seen that the solution of the linear differential equation with constant coefficients generally resolves itself ultimately into the solution of a number of linear differential equations of the form

$$\frac{dy}{dx} - ay = X \quad . \quad . \quad . \quad . \quad . \quad (335)$$

the particular integral of which, derived by use of an integrating factor, is given in equation 294 as

$$y = e^{ax}\int e^{-ax}X\,dx \quad . \quad . \quad . \quad . \quad . \quad (294\text{A})$$

This solution may be readily obtained by operational methods using the shifting theorem. Thus

$$\frac{1}{D-a} \cdot X = \frac{1}{D-a} \cdot 1 \cdot X = \frac{1}{D-a} \cdot \varepsilon^{ax} \cdot \varepsilon^{-ax} \cdot X$$

Using the shifting theorem (equation 328) this becomes

$$e^{ax}\frac{1}{D+a-a}e^{-ax}X = e^{ax}D^{-1}e^{-ax}X = e^{ax}\int e^{-ax}X\,dx$$

It should be noted that this operational form only gives the particular integral. As is shown later the complementary function can also be obtained by operational methods.

The Linear Differential Equation with Constant Coefficients.

The general form (equation 263) may be written

$$F(D) \cdot y = X \quad . \quad . \quad . \quad . \quad . \quad (335)$$

A first step in solving equation 263 was the determination of the complementary function which is the solution of

$$F(D)y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (336)$$

This may be included in equation 335 by rewriting it as

$$F(D) \cdot y = X + 0 \quad . \quad . \quad . \quad . \quad . \quad (335\text{A})$$

and the complete solution is then given as

$$y = F^{-1}(D) \cdot X + F^{-1}(D) \cdot 0 \quad . \quad . \quad . \quad (337)$$

The first term on the right gives the particular integral, the second the complementary function.

Operations upon Zero—the Complementary Function.

In its simplest form $F^{-1}(D)$ may occur as

$$F^{-1}(D) = (D - m_1)^{-1} \,.\, (D - m_2)^{-1} \,.\, (D - m_3)^{-1} \quad . \quad . \quad (338)$$

conveniently written as

$$F^{-1}(D) = \prod_{r=1,\,2\,\ldots} (D - m_r)^{-1} \quad . \quad . \quad . \quad . \quad (339)$$

or, splitting into partial fractions

$$F^{-1}(D) = \sum_{r=1,\,2\,\ldots} a_r (D - m_r)^{-1} \quad . \quad . \quad . \quad (340)$$

The operation of this upon zero resolves itself into the sum of a number of operations of the form $a(D - m)^{-1}$ upon zero. Now $(D - m)^{-1} \,.\, 0$ may be written as

$$(D - m)^{-1} e^{mx} \,.\, e^{-mx} \,.\, 0 = e^{mx} D^{-1} \,.\, e^{-mx} \,.\, 0$$

But $e^{-mx} \,.\, 0 = 0$ and therefore

$$(D - m)^{-1} \,.\, 0 = e^{mx} \,.\, D^{-1} \,.\, 0 = Ce^{mx}$$

the solution previously obtained.

In its most general form, however, $F^{-1}(D)$ may occur as

$$F^{-1}(D) = D^{-n} \,.\, (D - m_1)^{-k} \,.\, (D - m_2)^{-l} \quad . \quad . \quad . \quad (341)$$

It is in this case that classical methods are cumbersome. Resolving 341 into partial fractions indicates that solutions are required to

$$D^{-n} \,.\, 0 \text{ and } (D - m)^{-k} \,.\, 0$$

Considering first $D^{-n} \,.\, 0$

$$D^{-1} \,.\, 0 = \int 0 \, dx = C_1$$

$$D^{-2} \,.\, 0 = \int C_1 dx = C_2 + C_1 x$$

$$D^{-3} \,.\, 0 = \int (C_2 + C_1 x) dx = C_3 + C_2 x + C_1 \frac{x^2}{2}$$

$$D^{-n} \,.\, 0 = B_0 + B_1 x + B_2 x^2 \ldots B_{n-1} x^{n-1} = \sum_{r=0}^{n-1} B_r x^r \quad (342)$$

Again $(D - m)^{-k} \,.\, 0 = (D - m)^{-k} \,.\, e^{mx} \,.\, e^{-mx} \,.\, 0$

$$= e^{mx} \,.\, D^{-k} \,.\, e^{-mx} \,.\, 0$$

$$= e^{mx} \,.\, D^{-k} \,.\, 0$$

or $$(D - m)^{-k} \,.\, 0 = e^{mx} \sum_{r=0}^{k-1} B_r x^r \quad . \quad . \quad . \quad . \quad . \quad . \quad (343)$$

This is the form given on p. 153, determined here by a more satisfactory method.

An important case constantly recurring is $(D^2 + a^2)^{-1} \cdot 0$. This may be written as

$$\frac{1}{2ja}\left(\frac{1}{D - ja} - \frac{1}{D + ja}\right) \cdot 0 = k_1 e^{jax} + k_2 e^{-jax}$$

or

$$(D^2 + a^2)^{-1} \cdot 0 = A \cos ax + B \sin ax \quad . \quad . \quad (344)$$

Another important practical case is $(D^2 + aD + b)^{-1} \cdot 0$ which may be written $(D - m_1)^{-1}(D - m_2)^{-1} \cdot 0$ where

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \quad m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

and the solution if $a^2 > 4b$ is

$$(D^2 + aD + b)^{-1} \cdot 0 = Ae^{\frac{-a+\sqrt{a^2-4b}}{2}x} + Be^{\frac{-a-\sqrt{a^2-4b}}{2}x} \quad (345)$$

If however $a^2 < 4b$ it is more convenient to write the solution as

$$(D^2 + aD + b)^{-1} \cdot 0$$

$$= e^{-\frac{a}{2}x}\left[A \cos \sqrt{b - \frac{a^2}{4}}x + B \sin \sqrt{b - \frac{a^2}{4}}x\right] \quad (346)$$

If $a^2 = 4b$ then

$$(D^2 + aD + b)^{-1} \cdot 0 = e^{-\frac{a}{2}x}(A + Bx) \quad . \quad . \quad (347)$$

Inverse Operations upon Unity.

Cases have already arisen where it is necessary to carry out an inverse operation upon unity (for instance in obtaining equation 330). The method adopted was to write $1 = e^{0x}$ and to use equation 295.

$$F(D) \cdot e^{0x} = F(0)$$

and

$$F^{-1}(D) \cdot e^{0x} = F^{-1}(0)$$

Note that e^{0x} is here the subject, the operator is $F(D)$ or $F^{-1}(D)$ or

$$\left.\begin{aligned} F(D) \cdot 1 &= F(0) \\ \text{and} \quad F^{-1}(D) \cdot 1 &= F^{-1}(0) \end{aligned}\right\} \quad . \quad . \quad . \quad (348)$$

Equation 348 is useless if $F^{-1}(0)$ is infinite. This occurs if

D or D^n is a factor of $F(D)$. In this case one may write

$$F(D) = D^n\phi(D)$$

$$F^{-1}(D) \,.\, 1 = D^{-n}\frac{1}{\phi(D)} \,.\, 1 = D^{-n}\frac{1}{\phi(0)} \,.\, 1$$

$$= \frac{1}{\phi(0)} D^{-n} \,.\, 1$$

Immediately below it is shown that $D^{-n} \,.\, 1 = \frac{x^n}{\underline{|n}}$ and so

$$F^{-1}(D)1 = \frac{x^n}{\phi(0)\underline{|n}} \quad . \quad . \quad . \quad . \quad . \quad (349)$$

The case of $D^{-n} \,.\, 1$ is simple

$$D^{-1} \,.\, 1 = \int dx = x$$

$$D^{-2} \,.\, 1 = \int x\, dx = \frac{x^2}{2}$$

and

$$D^{-n} \,.\, 1 = \frac{x^n}{\underline{|n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (350)$$

Examples.

1. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2e^{3x}$

$$y = \frac{1}{(D-1)^2} \,.\, 0 + \frac{1}{(D-1)^2} e^{3x}x^2$$

$$= e^x(a + bx) + e^{3x}\frac{1}{(D+2)^2}x^2$$

$$= e^x(a + bx) + \frac{e^{3x}}{4}(1 - D + \tfrac{3}{4}D^2 \ . \ . \ .)x^2$$

$$= e^x(a + bx) + \frac{e^{3x}}{4}(x^2 - 2x + \tfrac{3}{2})$$

2. Solve $\dfrac{d^2x}{dt^2} + \omega^2x = k \cos \omega t$

$$x = \frac{1}{p^2 + \omega^2} \,.\, 0 + \frac{1}{p^2 + \omega^2} k \cos \omega t$$

or

$$x = A \cos \omega t + \mathrm{B} \sin \omega t + \frac{kt}{2\omega} \sin \omega t$$

from equations 344 and 334.

3. Solve $\frac{dy}{dx} - ay = \frac{1}{x}$

$$y = \frac{1}{D-a} \cdot 0 + \frac{1}{D-a} \cdot \frac{1}{x}$$

$$= ce^{ax} + \frac{1}{D-a} e^{ax} e^{-ax} \cdot \frac{1}{x}$$

$$= ce^{ax} + e^{ax} \frac{1}{D} \cdot \frac{1}{xe^{ax}}$$

$$= ce^{ax} + e^{ax} \int \frac{1}{xe^{ax}} dx$$

This is not directly integrable as it stands, but it is easily obtained as an infinite series either by expanding e^{-ax}, dividing by x, and then integrating term by term, or by expressing $\frac{1}{D-a}$ as a series in ascending powers of D operating upon $\frac{1}{x}$.

CHAPTER XIII

EXAMPLES INVOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In this chapter a number of examples will be considered involving the differential equations dealt with in immediately preceding chapters. The linear differential equation arises continually in problems dealing with vibrations, free or forced. These vibrations or oscillations may be mechanical or electrical or often both.

Series Circuit containing Resistance, Inductance and Capacitance.

If i be the current in the circuit of Fig. 76 then

$$\varepsilon = iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad . \quad . \quad . \quad . \quad (351)$$

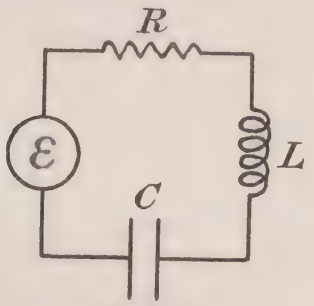


FIG. 76.

Differentiating and dividing by L this becomes

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\frac{d\varepsilon}{dt} \quad . \quad . \quad . \quad (352)$$

(i) if ε is constant then $\dfrac{d\varepsilon}{dt} = 0$ and the equation degenerates to

$$\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right)i = 0 \quad . \quad . \quad . \quad . \quad (353)$$

where $p \equiv \frac{d}{dt}$, and the solution is, from equations 345, 346 and 347

(*a*) if $\frac{R^2}{4L^2} > \frac{1}{LC}$

$$i = Ae^{m_1 t} + Be^{m_2 t} \quad . \quad . \quad . \quad . \quad (354)$$

where $$m_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

and $$m_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

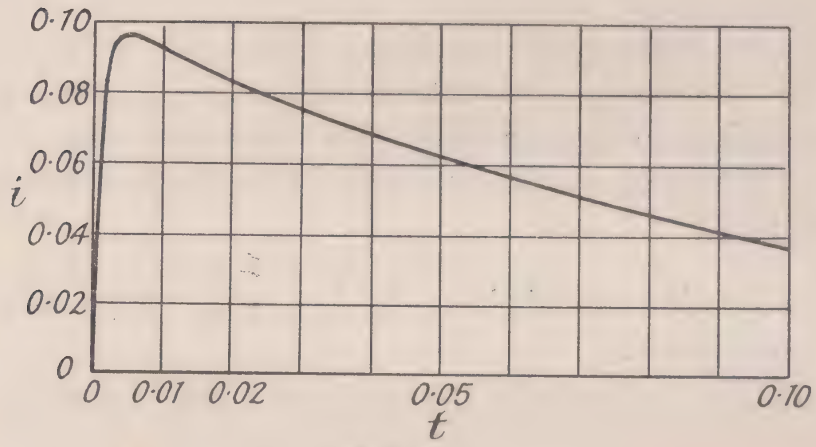


FIG. 77.

One form of this curve is given in Fig. 77.

(*b*) if $\frac{R^2}{4L^2} < \frac{1}{LC}$

$$i = \varepsilon^{mt}(A \cos \omega' t + B \sin \omega' t) \quad . \quad . \quad . \quad (355)$$

where $$m = -\frac{R}{2L} \text{ and } \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

One form of this curve is shown in Fig. 78—a decrescent sinoidal oscillation. The time of oscillation is $t' = 2\pi/\omega'$ and so, per cycle, the amplitude at all corresponding points in the oscillation decreases in the ratio $e^{\frac{R}{2L}t'} = e^{\frac{R}{2L}\cdot\frac{2\pi}{\omega'}} = e^{\frac{\pi R}{\omega' L}}$. The logarithm of this ratio, namely $\pi R/\omega' L$, is called the *logarithmic decrement*,

(*c*) if $\frac{R^2}{4L^2} = \frac{1}{LC}$

$$i = e^{mt}(A + Bt) \quad . \quad . \quad . \quad . \quad . \quad (356)$$

where $m = -R/2L$.

One form of this curve is shown in Fig. 79. The circuit is said to be critically damped. The final condition is approached as rapidly as possible without overswinging. The integration constants are determined by the initial conditions. This will be clear from the three practical cases considered. In each case it will be assumed that the condenser is charged to 100 volts

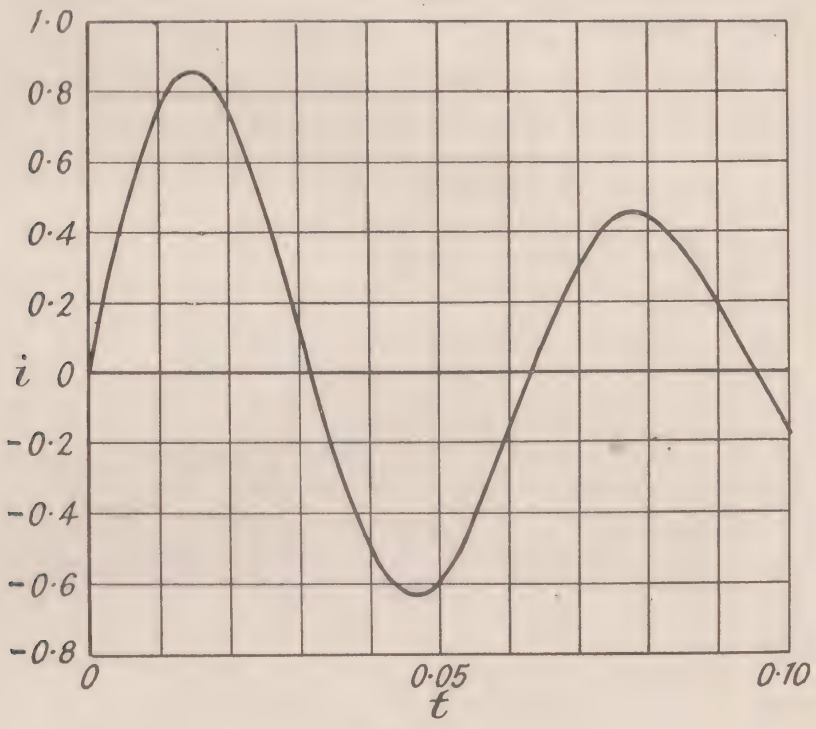


Fig. 78.

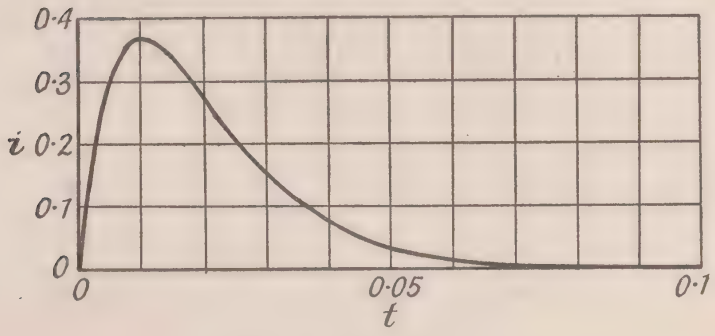


Fig. 79.

and that at $t = 0$, ε is shortcircuited; $L = 1$ henry; $C = 100$ μF and R is successively in the three cases (a') 1000 ohms, (b') 20 ohms, (c') 200 ohms. In each case at $t = 0$, $i = 0$ and $\frac{1}{C}\int i\,dt = 100$. Hence at $t = 0$, from equation 351, $0 = 0 + L\frac{di}{dt} + 100$ or $\frac{di}{dt} = -100$ amperes per

second. In equation 351 a positive current indicates that the condenser is charging. The negative sign for $\frac{di}{dt}$ is therefore natural since the condenser commences to *discharge* at $t = 0$.

(a') $R = 1000$, $R/2L = 500$, $1/LC = 10^4$

$$\frac{R^2}{4L^2} - \frac{1}{LC} = 24 \times 10^4, \quad \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = 489{\cdot}9$$

$$m_1 = -10{\cdot}1, \quad m_2 = -990$$

At $t = 0$, $i = 0$, $A + B = 0$, $B = -A$.

Again $\frac{di}{dt} = m_1 A e^{m_1 t} + m_2 B e^{m_2 t}$.

At $t = 0$, $m_1 A + m_2 B = -100$.

$$(m_1 - m_2)A = -100 = 980A$$

$$A = -0{\cdot}102, \quad B = +0{\cdot}102$$

$$i = -0{\cdot}102(e^{-10{\cdot}1t} - e^{-990t})$$

This is the curve plotted in Fig. 77.

(b') $R = 20$, $R/2L = 10$, $1/LC = 10^4$

$$\frac{1}{LC} - \frac{R^2}{4L^2} = 9{,}900, \quad \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 99{\cdot}5 = \omega'$$

$$m = -10$$

At $t = 0$, $i = 0$, $A = 0$ and so

$$i = Be^{mt} \sin \omega' t$$

then $$\frac{di}{dt} = mBe^{mt} \sin \omega' t + \omega' Be^{mt} \cos \omega' t$$

At $t = 0$, $\omega' B = -100$, $B = -1{\cdot}005$

$$i = -1{\cdot}005e^{-10t} \sin 99{\cdot}5t$$

This is the curve plotted in Fig. 78.

(c') $R = 200$, $R/2L = 100 = -m$

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

$$i = e^{mt}(A + Bt)$$

when $t = 0$, $i = 0$, $A = 0$

$$i = Bte^{mt}$$

$$\frac{di}{dt} = Be^{mt} + mBte^{mt}$$

At $t = 0$, $\frac{di}{dt} = -100 = B$

$$i = -100te^{-100t}$$

This is the curve plotted in Fig. 79.

(ii) When the impressed e.m.f. is alternating, say $\varepsilon = E_m \sin(\omega t + \alpha)$, equation 352 becomes

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{\omega E_m}{L}\cos(\omega t + \alpha) \quad . \quad . \quad (357)$$

and the operational equation is

$$\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right)i = 0 + \frac{\omega E_m}{L}\cos(\omega t + \alpha) \quad . \quad (358)$$

The complementary function has the same form as before although the values of the constants of integration are affected by the value of α (the instant in the applied cycle of e.m.f. at which connection is made). The particular integral is given by

$$i = \frac{\omega E_m}{L} \cdot \frac{1}{p^2 + \frac{R}{L}p + \frac{1}{LC}}\cos(\omega t + \alpha) \quad . \quad . \quad (359)$$

$$= \frac{\omega E_m}{L} \cdot \frac{1}{\left(\frac{1}{LC} - \omega^2\right) + \frac{R}{L}p}\cos(\omega t + \alpha)$$

(from equation 332, p. 171).

$$= \frac{\omega E_m}{L} \cdot \frac{\left(\frac{1}{LC} - \omega^2\right) - \frac{R}{L}p}{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2}\cos(\omega t + \alpha)$$

After reduction this gives

$$i = \frac{E_m}{R^2 + X^2}[R\sin(\omega t + \alpha) - X\cos(\omega t + \alpha)]$$

where $X = \omega L - \dfrac{1}{\omega C}$

or
$$i = \frac{E_m}{\sqrt{R^2 + X^2}}\sin(\omega t + \alpha - \phi) \quad . \quad . \quad (360)$$

where $\tan\phi = \dfrac{X}{R}$

It is seen that this is an alternating current whose magnitude is given by $I = \dfrac{E}{Z}$ where $Z = \sqrt{R^2 + X^2}$; Z is the impedance.

The phase of this current lags by ϕ behind the phase of the voltage. The complete solution is

$$i = \frac{E_m}{Z} \sin(\omega t + \alpha - \phi) + (354), (355), \text{ or } (356) \quad (361)$$

according as $\frac{R^2}{4L^2} >$ or $<$ or $= \frac{1}{LC}$ respectively.

The component 354, 355 or 356 dies out after a short time (transient) and then the solution is the permanent component given by equation 360. This component may assume large values when $X = 0$ if R is small (resonance).

An example will illustrate the effect upon the transient of the instant at which contact is made. The constants assumed are $L = 1$ henry, $C = 100$ μF, $R = 20$ ohms, $\omega = 200$, $E_m = 100$ volts. $\alpha = (a)$ 0°, (b) 90°. The circuit is dead at the instant of closing the switch. The general form of the solution is

$$i = e^{mt}(A \cos \omega' t + B \sin \omega' t) + \frac{E_m}{Z} \sin(\omega t + \alpha - \phi) \quad (362)$$

$m = -10$, $\omega' = 99{\cdot}5$, $\omega L = 200$, $\frac{1}{\omega C} = 50$, $X = 150$,

$Z = 151{\cdot}33$, $\tan \phi = 7{\cdot}5$, $\phi = 82°{\cdot}4$.

$i = e^{-10t}(A \cos 99{\cdot}5t + B \sin 99{\cdot}5t) + 0{\cdot}661 \sin(200t + \alpha - 82°{\cdot}4)$

Also at $t = 0$, $\varepsilon = L\frac{di}{dt}$

$$\frac{di}{dt} = -10e^{-10t}(A \cos 99{\cdot}5t + B \sin 99{\cdot}5t) + \varepsilon^{-10t}(-99{\cdot}5\, A \sin 99{\cdot}5t + 99{\cdot}5\, B \cos 99{\cdot}5t) + 0{\cdot}661 \times 200 \cos(200t + \alpha - 82°{\cdot}4)$$

At $t = 0$, $\frac{di}{dt} = -10A + 99{\cdot}5B + 132{\cdot}2 \cos(\alpha - 82°{\cdot}4)$

(a) At $t = 0$, $i = 0$

$$0 = A - 0{\cdot}661 \sin 82°{\cdot}4\,; \quad A = 0{\cdot}655$$

when $t = 0$, $\varepsilon = 0$, $\frac{di}{dt} = 0$

$$0 = -10A + 99{\cdot}5B + 132{\cdot}2 \cos 82°{\cdot}4$$
$$= -10A + 99{\cdot}5B + 17{\cdot}5, \quad B = -0{\cdot}110$$

So that in this case the transient term is

$$i = e^{-10t}(0{\cdot}655 \cos 99{\cdot}5t - 0{\cdot}110 \sin 99{\cdot}5t)$$

(b) At $t = 0$, $i = 0$

$$0 = A + 0{\cdot}661 \sin 7°{\cdot}6, \quad A = -0{\cdot}0875$$

When $t = 0$, $\varepsilon = 100$, $\dfrac{di}{dt} = 100$ and so

$$100 = -10A + 99{\cdot}5B + 132{\cdot}2 \cos 7°{\cdot}6$$
$$B = -0{\cdot}321 \text{ and the transient term is}$$
$$i = e^{-10t}(-0{\cdot}0875 \cos 99{\cdot}5t - 0{\cdot}321 \sin 99{\cdot}5t)$$

In this case the magnitude of the transient term is considerably less and its phase is very different.

Synchronous Spark Transmitter.

A case of considerable interest arises when $\omega = \dfrac{1}{\sqrt{LC}}$ and the switch is closed with the circuit dead at $\alpha = 0$, the value of R being small. This is the arrangement adopted in the radio-telegraphic synchronous spark transmitter. At zero on the low frequency voltage wave the condenser has no charge. The circuit is tuned to resonance at low frequency. As the low frequency voltage wave grows as ωt proceeds from 0 to π the condenser charges. It will be shown that the conditions are such that at $\omega t = \pi$ the charging current has fallen to zero and the condenser voltage is at its maximum. At this instant the condenser is connected to the radio-frequency circuit and is discharged almost instantaneously (considered from the period of the low frequency alternating voltage). Power is not poured in from the low frequency power circuit because at this instant the voltage is zero ($\omega t = \pi$) ; before the low frequency voltage has risen appreciably the contacts connecting the radio-frequency circuit to the condenser have opened. The sequence is then repeated in the next half-cycle.

The conditions will be worked out in two different ways. Assuming the result of equation 361 one may write

$$i = \frac{E_m}{R} \sin \omega t + e^{mt}(A \cos \omega' t + B \sin \omega' t) \quad . \quad (363)$$

since $\alpha = 0$ and $\phi = 0$ and $Z = R$.

$$\frac{di}{dt} = \frac{\omega E_m}{R} \cos \omega t + me^{mt}(A \cos \omega' t + B \sin \omega' t) + \omega' e^{mt}(-A \sin \omega' t + B \cos \omega' t) \quad (364)$$

when $t = 0$, $i = 0$, $E = 0$ and so $\dfrac{di}{dt} = 0$ and then from

equations 363 and 364 one obtains $0 = 0 + A$ or $A = 0$

$$0 = \frac{\omega E_m}{R} + Am + \omega' B \text{ or } B = -\frac{\omega}{\omega'}\frac{E_m}{R}$$

and

$$i = \frac{E_m}{R}\left(\sin \omega t - e^{mt}\frac{\omega}{\omega'}\sin \omega' t\right) \quad . \quad . \quad . \quad (365)$$

This expression for i is indeterminate when $R \longrightarrow 0$. Writing

$$m = -\frac{R}{2L} = -b$$

$$\omega' = \sqrt{\omega^2 - b^2}, \quad e^{mt} = e^{-bt} = 1 - bt + \frac{b^2t^2}{2} \quad . \quad . \quad . \quad .$$

Therefore ω' differs from ω by the second order of small quantities, while ε^{-bt} differs from unity by the first order of small quantities and therefore one may write

$$\underset{R \longrightarrow 0}{i} = \frac{E_m}{R}\sin \omega t(1 - e^{-bt}) \text{ or}$$

$$\underset{R \longrightarrow 0}{i} = \frac{E_m}{R}\sin \omega t \cdot \frac{R}{2L}t = \frac{E_m}{2L}t \sin \omega t \quad . \quad . \quad (366)$$

This result may be obtained much more neatly directly. When $R = 0$ and $\alpha = 0$ the operational equation 358 reduces to (since $\frac{1}{LC} = \omega^2$)

$$(p^2 + \omega^2)i = \frac{\omega E_m}{L}\cos \omega t$$

or

$$i = (p^2 + \omega^2)^{-1} \cdot 0 + \frac{\omega E_m}{L} \cdot (p^2 + \omega^2)^{-1} \cdot \cos \omega t \quad (367)$$

From equation 344, $(p^2 + \omega^2)^{-1} \cdot 0 = A \cos \omega t + B \sin \omega t$ and from equation 334, $(p^2 + \omega^2)^{-1} \cdot \cos \omega t = \frac{t}{2\omega}\sin \omega t$ and so the complete solution is

$$i = A \cos \omega t + B \sin \omega t + \frac{E_m}{2L}t \sin \omega t \quad . \quad . \quad (368)$$

When $t = 0$, $i = 0$ and so $A = 0$.

Again $\left(\frac{di}{dt}\right)_{t=0} = \omega B$ and therefore $B = 0$ and the solution is

$$i = \frac{E_m}{2L} t \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (366)$$

The form of this curve is shown in Fig. 80.

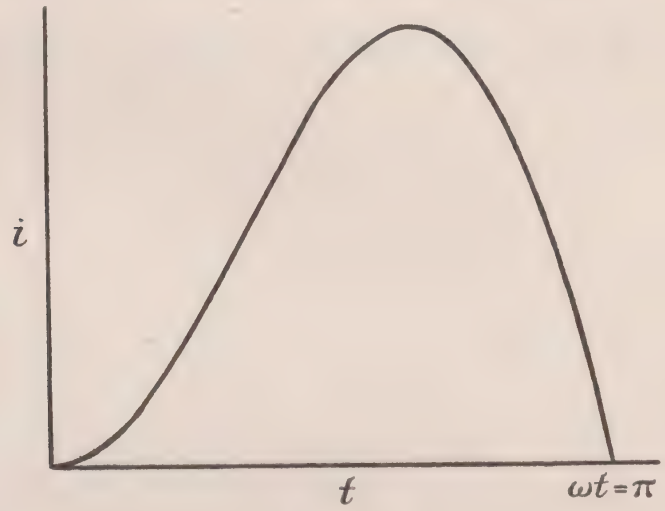


FIG. 80.

The voltage on the condenser is

$$\frac{1}{C}\int i\, dt = \frac{E_m}{2LC}\left(\frac{1}{\omega^2} \sin \omega t - \frac{t}{\omega} \cos \omega t + K\right) = v \text{ say.}$$

When $t = 0$, $v = 0$ and so $K = 0$ and since $\frac{1}{LC} = \omega^2$

$$v = \frac{E_m}{2}(\sin \omega t - \omega t \cos \omega t) \quad . \quad . \quad . \quad (369)$$

when $\omega t = \pi$, equations 366 and 369 give

$$i = 0 \text{ and } v = \frac{\pi}{2} E_m \quad . \quad . \quad . \quad . \quad (370)$$

The condenser is charged to $\frac{\pi}{2}$ times the maximum value of the A.C. potential or $\frac{\pi}{\sqrt{2}}$ or 2·22 times the R.M.S. value.

The Damped Vibration considered Generally.

The general form of the equation for many damped vibrations is (one particular case has been considered and others are given below)

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f(t) \quad . \quad . \quad . \quad . \quad (371)$$

where y is some co-ordinate defining the vibration. Thus if y is the current in a circuit of resistance R, inductance L and capacitance C, $a = \frac{R}{L}$, $b = \frac{1}{LC}$ and $f(t) = \frac{1}{L}\frac{de}{dt}$ (equation 352). The solutions obtained for equation 352 may therefore be transferred to equation 371 with these substitutions. Thus if $\frac{a^2}{4} > b$ the motion is non-oscillatory and the transient motion is given by

$$y = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t} \quad . \quad . \quad . \quad . \quad (372)$$

where $m_1 = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}$ and $m_2 = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - b}$.

If $\frac{a^2}{4} < b$ the transient motion is oscillatory and its frequency is $\omega' = \sqrt{b - \frac{a^2}{4}}$. The logarithmic decrement is $\frac{\pi a}{\omega'}$. When the damping is small this is nearly equal to $\pi\frac{a}{\sqrt{b}}$.

If $\frac{a^2}{4} = b$ the system is critically damped.

Resonance occurs when sinoidal disturbing influences (forces, or electromotive forces, etc.) are applied whose angular frequency is $\omega = \sqrt{b}$.

For mechanical motions the equation is of the form of equation 371 when the frictional, or viscous (or generally non-reversible) forces acting are proportional to the velocity, the reversible forces (other than inertial) being proportional to the displacement. If x is the displacement of a body from its position of equilibrium, its velocity in the direction of x increasing is $\frac{dx}{dt}$ and its acceleration is $\frac{d^2x}{dt^2}$. If the displacement is resisted by an elastic force $= \lambda x$ say, and the resistance to motion is proportional to the velocity and is, say, $\mu\frac{dx}{dt}$ then if the mass is m the equation of motion is

$$P = m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \lambda x \quad . \quad . \quad . \quad . \quad (373)$$

where P is the force acting. If the motion is angular about a

fixed axis, the body having a moment of inertia K, then under a torque T the equation of motion is

$$T = K\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \beta\theta \quad . \quad . \quad . \quad . \quad (374)$$

where α is the resisting torque due to friction or damping for unit angular velocity and β is the elastic resisting torque per unit angle of displacement.

An application of equation 373 is afforded by the following example.

" Show that in a simple system making free vibrations with viscous damping the ratio of the frequency of vibration to the frequency when there is no damping is given by the expression $\sqrt{1 - \frac{c^2}{4km}}$, where c is the damping coefficient (force per unit velocity), k is the elastic stiffness of the system, and m is the mass of the vibrating body.

" Explain the meaning of critical damping and show that this condition is reached when $c = 2m\omega$, where ω is the angular frequency of the system without damping." (Lond. Univ. B.Sc. Eng. Ext., 1937, Advanced Theory of Machines and Machine Design, I.4.)

The equation of motion is

$$P = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

or

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{P}{m}$$

When there is no damping the frequency of vibration is equal to the resonant frequency. The resonant angular frequency is $\omega = \sqrt{\frac{k}{m}}$. The frequency of vibration with damping is $\omega' = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$ whence

$$\omega'/\omega = \sqrt{1 - \frac{c^2}{4km}}$$

The motion is critically damped when $\frac{k}{m} = \frac{c^2}{4m^2}$ or $\omega^2 = \frac{c^2}{4m^2}$ giving $c = 2m\omega$.

Moving Coil Galvanometers,

Moving coil galvanometers may be used as deflecting instruments (D'Arsonval type), ballistic instruments or vibration instruments. Basically one has a coil suspended in a magnetic field. The motion is defined by equation 374. If h = height of coil, w = width of coil, i = current, H = strength of magnetic field and N = number of turns in the coil, then (working in C.G.S. units)

Force on one side of coil = $NhHi$ dynes

Torque = $NhHiw = NHAi = Gi$ dyne cms.

where A = area of coil and $G = NHA$.

When the coil moves an e.m.f. is generated in it. If the velocity of one side of the coil is u, then the e.m.f. generated in one side is $NhHu$ and in the whole coil $2NhHu$. But $u = \frac{w}{2}\frac{d\theta}{dt}$ and so the e.m.f. generated in the coil by its motion is $NwhH\frac{d\theta}{dt} = G\frac{d\theta}{dt}$.

These two equations are fundamental

$$\text{Torque} = Gi \quad . \quad . \quad . \quad . \quad . \quad . \quad (375)$$

$$\text{e.m.f.} = G\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad (376)$$

This e.m.f. is of course in opposition to the e.m.f. deflecting the coil. If one assumes that the inductance of the coil is negligible (a more exact theory is given later in connection with the moving coil speaker) then, if the applied e.m.f. be e and the resistance of the circuit be R, the current is

$$i = \frac{1}{R}\left(e - G\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (377)$$

and the equation of motion is

$$\frac{G}{R}\left(e - G\frac{d\theta}{dt}\right) = K\frac{d^2\theta}{dt^2} + \alpha\frac{d\theta}{dt} + \beta\theta$$

or

$$\frac{Ge}{R} = K\frac{d^2\theta}{dt^2} + \left(\alpha + \frac{G^2}{R}\right)\frac{d\theta}{dt} + \beta\theta \quad . \quad . \quad . \quad (378)$$

It is seen that in addition to the damping due to air resistance, etc., represented by the coefficient α, there is electromagnetic damping represented by the coefficient $\frac{G^2}{R}$. Often α is much

smaller than $\frac{G^2}{R}$ and the instrument may be critically damped by adjustment of R. The damping is critical when

$$\left(\alpha + \frac{G^2}{R}\right)^2 = 4K\beta \quad . \quad . \quad . \quad . \quad . \quad (379)$$

The *constants* of the instrument may be simply expressed. The sensitivity to a steady current is given by putting $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ equal to zero when one obtains

$$\frac{\theta}{i} = \frac{G}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (380)$$

The undamped period is defined by $\omega = \sqrt{\frac{\beta}{K}} = 2\pi f$

giving

$$\frac{1}{f} = 2\pi\sqrt{\frac{K}{\beta}} \quad . \quad . \quad . \quad . \quad . \quad (381)$$

When the damping is wholly electromagnetic the resistance required to give critical damping is (from equation 379)

$$R = \frac{G^2}{2\sqrt{K\beta}} \quad . \quad . \quad . \quad . \quad . \quad (382)$$

Examples.

1. " What is meant by critical damping and to what extent should this condition be approached in the case of an ordinary moving coil type of instrument ?

" The coil of a moving coil galvanometer has 300 turns and is suspended in a uniform magnetic field of 1000 lines per sq. cm. by a phosphor bronze strip of which the torsion constant is 2 dyne cm. per radian. The coil is 2 cm. wide and $2\frac{1}{2}$ cm. high with a moment of inertia of 1·5 grm. cm.^{2}

" If the galvanometer resistance is 200 ohms, calculate the value of the resistance which, when connected across the galvanometer terminals, will give critical damping. Assume the damping to be entirely electromagnetic." (Lond. Univ. B.Sc. Eng. Ext., 1935, Electrical Measurements and Measuring Instruments, II. 8.)

Critical damping is such that the motion is just rendered non-oscillatory. If the damping is much greater than this the final reading is approached but slowly ; if the damping is much less the spot vibrates backwards and forwards about its final

value and it is some time before a reading can be obtained. The optimum condition requires damping which approximates to critical. Some workers prefer damping slightly less than critical, the small reverse motion being a definite indication that the full deflection has been reached.

$G=NHA=300\times1000\times5=1{\cdot}5\times10^6$

$K=1{\cdot}5,\ \beta=2,\ 4K\beta=12,\ \dfrac{G^4}{R^2}=12$

$\dfrac{G^2}{R}=\sqrt{12},\ R=\dfrac{G^2}{\sqrt{12}}=\dfrac{2{\cdot}25\times10^{12}}{\sqrt{12}}=0{\cdot}65\times10^{12}$ C.G.S. units

$=650$ ohms. The added resistance is 450 ohms.

2. " A moving coil galvanometer has a sensitivity of 4 cm per micro-ampere with the scale 1 metre distant, and the time of free oscillation is 2·8 seconds. If the galvanometer is dead beat when the total circuit resistance (coil and external circuit) is 2500 ohms, find the moment of inertia of the moving system. Prove any formula used." (Lond. Univ. B.Sc. Eng. Ext., 1934, Electrical Measurements and Measuring Instruments, I. 7.)

Angle turned through by coil = half angle of deflection. For 1 μA, $= 10^{-7}$ C.G.S. units, $\theta = 1/50$; $\theta/i = 2\times10^5$; $G/\beta = 2\times10^5$; $2{\cdot}8 = 2\pi\sqrt{\dfrac{K}{\beta}}$; $\dfrac{K}{\beta} = 0{\cdot}199$; $\dfrac{G^4}{R^2}=4K\beta$; $R = 2{\cdot}5\times10^{12}$; $R^2 = 6.25\times10^{24}$; $G^4 = 25\times10^{24}K\beta$.

The equations from which K has to be determined are $G = 2\times10^5\beta$; $G^4 = 25\times10^{24}K\beta$; $\dfrac{K}{\beta} = 0{\cdot}199$. From the first equation $G^4 = 16\times10^{20}\beta^4$ so that

$$\frac{K}{\beta^3} = 0{\cdot}64\times10^{-4};\quad \frac{K}{\beta} = 0{\cdot}199 \text{ giving } \beta^2 = 0{\cdot}311\times10^4$$

$\beta = 55{\cdot}8$ and $K = 11{\cdot}1$.

The examples given refer particularly to the D'Arsonval type of galvanometer. The vibration galvanometer is used as a detecting instrument in A.C. circuits. The movement is very light and the suspension strained to the point that the natural frequency is equal to that of the system upon which it is being used. If the damping is then small the steady vibration caused by quite a small current at the resonant frequency is considerable but the instrument is comparatively insensitive to other

frequencies (harmonics) present. The solution of the relevant equation is of course obtained in the same way as the solution of equation 357. Rewriting equation 378 as

$$K\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + \beta\theta = Gi$$

then if $i = I_m \cos \omega t$ the particular integral is

$$\theta = \frac{GI_m}{\sqrt{(\beta - K\omega^2)^2 + a^2\omega^2}} \cos(\omega t - \phi) \quad . \quad . \quad (383)$$

where $\tan \phi = a\omega/(\beta - K\omega^2)$.

At the resonant frequency this becomes

$$\theta = \frac{GI_m}{a\omega} \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (384)$$

When the damping is small $\beta - K\omega^2 > > a\omega$ except near the resonant frequency and so very small deflections are obtained when the frequency of the current is much different from the resonant frequency.

The Ballistic Galvanometer.

The ballistic galvanometer is used for measuring small *quantities* of electricity. The time of discharge is small compared with the natural period of the instrument. To secure this the moment of inertia is usually considerable and the controlling torque small. Damping is reduced to a minimum. The complete theory of the instrument cannot be expressed unless the nature of the discharge is known. An indication of the errors which may arise can be obtained by assuming a simple form of discharge such as a constant current for a short time. If i is constant the complete solution of equation 378 is

$$\theta = \varepsilon^{-\frac{a}{2K}t}\left[A \cos \omega t + B \sin \omega t\right] + \frac{Gi}{\beta} \text{ where } \omega = \sqrt{\frac{\beta}{K} - \frac{a^2}{4K^2}}$$

If $\theta = 0$, and $\frac{d\theta}{dt} = 0$ when $t = 0$ then it is easy to show that

$$\theta = \frac{Gi}{\beta}\left[1 - e^{-\frac{a}{2K}t}\left(\cos \omega t + \frac{a}{2\omega K} \sin \omega t\right)\right] \quad . \quad (385)$$

and

$$\frac{d\theta}{dt} = \frac{Gi}{K\omega} e^{-\frac{a}{2K}t} \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (386)$$

If t is short then to the first order of small quantities $\theta = 0$ and $\frac{d\theta}{dt} = \frac{Git}{K}$

or

$$K\frac{d\theta}{dt} = GQ \quad . \quad . \quad . \quad . \quad . \quad . \quad (387)$$

where Q is the quantity of electricity discharged through the instrument.

This equation might have been obtained directly from mechanical considerations. Before the instrument has had time to deflect appreciably the only torque acting is that due to the current and is equal to Gi, and so

$$K\frac{d\theta}{dt} = G\int i\,dt = GQ$$

Equations 385 and 386 however enable a check to be made upon possible errors in assuming that the discharge is instantaneous. The subsequent motion is given by

$$\theta = e^{-\frac{a}{2K}t}[A\cos\omega t + B\sin\omega t]$$

given that when $t = 0$, $\theta = 0$ and $\frac{d\theta}{dt} = \frac{GQ}{K}$

giving

$$\theta = \frac{GQ}{\omega K}e^{-\frac{a}{2K}t}\sin\omega t \quad . \quad . \quad . \quad . \quad (388)$$

So long as the damping is small, maximum readings in the positive direction are reached when $\omega t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$, etc., and in the negative direction when $\omega t = \frac{3\pi}{2}, \frac{7\pi}{2}$, etc. From a series of readings the deflection which would be obtained were there no damping is easily calculated. If there is no damping then equation 388 becomes

$$\theta = \frac{GQ}{\omega K}\sin\omega t \text{ where } \omega = \sqrt{\frac{\beta}{K}}$$

or

$$\theta = \frac{GQ}{\sqrt{\beta K}}\sin\sqrt{\frac{\beta}{K}}t \quad . \quad . \quad . \quad . \quad . \quad . \quad (389)$$

Example.

" Write a theory of the moving coil type of galvanometer used ballistically.

" Find the sensitivity of a ballistic galvanometer given the following data :

" Coil 400 turns ; 3 cm. long by 2 cm. wide ; moment of inertia about vertical axis 5·0 grm. cm.^{2}

" Suspension : torsional stiffness, 1·5 dyne cm. per radian.

" Radial magnetic field 800 lines per cm.^{2}

" Damping, negligible." (Lond. Univ. B.Sc. Eng. Ext., 1937, Electrical Measurements and Measuring Instruments, I. 4.)

The maximum value of θ is from equation 389

$$\theta_m = \frac{GQ}{\sqrt{\beta K}}$$

$$\beta = 1{\cdot}5 \quad K = 5 \quad G = NAH = 400 \times 6 \times 800 = 1{\cdot}92 \times 10^6$$

$$\frac{\theta_m}{Q} = \frac{1{\cdot}92 \times 10^6}{\sqrt{7{\cdot}5}} = 0{\cdot}70 \times 10^6$$

Q is in C.G.S. units or tens of coulombs. Hence $\theta = 0{\cdot}07$ radians per micro-coulomb or approximately 4° per micro-coulomb.

The natural period of this instrument is

$$2\pi\sqrt{\frac{K}{\beta}} = 2\pi\sqrt{\frac{5}{1{\cdot}5}} = 11{\cdot}4 \text{ secs.}$$

CHAPTER XIV

SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS; PRACTICAL EXAMPLES

So far the differential equations which have been considered have involved only one dependent and one independent variable. The most general equation is of course the partial differential equation in which the dependent variables are conditioned by more than one independent variable. This will call for consideration later. There is, however, an important class of equations occurring continually in electrical circuit theory in which there is only one independent variable but more than one dependent variable. The case which will be considered is that in which there are as many linear equations as dependent variables. From these, equations must be obtained in each of which only one dependent variable appears. One equation having been solved, more than one obvious method may appear for obtaining a solution for the other independent variables. The method may be illustrated by a simple example.

$$\text{Solve } \left.\begin{aligned} \frac{dx}{dt} - 7x + y &= 0 \\ \frac{dy}{dt} - 2x - 5y &= 0 \end{aligned}\right\}$$

Differentiating the second of these equations gives

$$\frac{d^2y}{dt^2} - 2\frac{dx}{dt} - 5\frac{dy}{dt} = 0$$

These three equations suffice for the elimination of x and $\frac{dx}{dt}$. Multiplying the third by 1, the second by -7, and the first by 2, and adding, one obtains

$$\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 37y = 0$$

From equation 346 this gives

$$y = e^{6t}(A \cos t + B \sin t)$$

$$\frac{dy}{dt} = 6e^{6t}(A \cos t + B \sin t) + \varepsilon^{6t}(-A \sin t + B \cos t)$$

or $$e^{6t}[(6A + B) \cos t + (6B - A) \sin t]$$

Substituting in the second of the original equations gives

$$2x = e^{6t}[(A + B) \cos t + (B - A) \sin t]$$

or $$x = \tfrac{1}{2}e^{6t}[(A + B) \cos t + (B - A) \sin t]$$

The work is somewhat simplified by using the symbol p for $\frac{d}{dt}$ and treating it as an algebraic multiplier. Thus the original equations may be written

$$(p - 7)x + y = 0$$
$$(p - 5)y - 2x = 0$$

Multiplying the former by 2 and the latter by $p - 7$ and adding gives

$$(p^2 - 12p + 37)y = 0$$

From this point the work is as before.

The Transformer.

Many examples of this nature occur in electrical circuit theory. Thus in the coupled circuits of Fig. 81, the equations

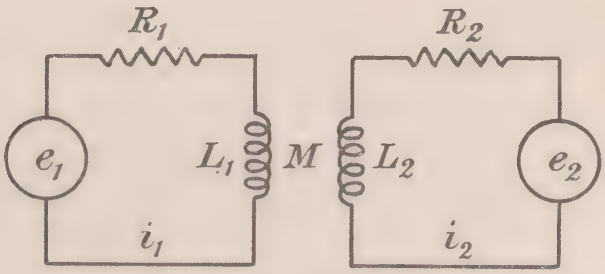


Fig. 81.

are

$$e_1 = i_1R_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \quad . \quad . \quad . \quad (390)$$

and

$$e_2 = i_2R_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} \quad . \quad . \quad . \quad (391)$$

conveniently written as

$$(pL_1 + R_1)i_1 + pMi_2 = e_1 \quad . \quad . \quad . \quad (390a)$$

and

$$(pL_2 + R_2)i_2 + pMi_1 = e_2 \quad . \quad . \quad . \quad (391a)$$

Multiplying the former by $pL_2 + R_2$ and the latter by pM and subtracting gives

$$(L_1L_2 - M^2)p^2i_1 + (L_1R_2 + L_2R_1)pi_1 + R_1R_2i_1 = L_2pe_1 - Mpe_2 + R_2e_1 \quad . \quad . \quad (392a)$$

or

$$(L_1L_2 - M^2)\frac{d^2i_1}{dt^2} + (L_1R_2 + L_2R_1)\frac{di_1}{dt} + R_1R_2i_1 = L_2\frac{de_1}{dt} - M\frac{de_2}{dt} + R_2e_1 \quad . \quad . \quad (392)$$

And similarly

$$(L_1L_2 - M^2)\frac{d^2i_2}{dt^2} + (L_1R_2 + L_2R_1)\frac{di_2}{dt} + R_1R_2i_2 = L_1\frac{de_2}{dt} - M\frac{de_1}{dt} + R_1e_2 \quad . \quad . \quad (393)$$

In most practical cases $e_2 = 0$. The equations will be solved when e_1 is a constant direct potential E. When $e_1 = E_m \cos \omega t$ interest is usually less in the transient than in the steady alternating currents in the two circuits. This problem is more easily solved by vectors. Its solution is given on p. 108.

When $e_1 = E$ and $e_2 = 0$ equation 392a reduces to

$$(p^2 + ap + b)i_1 = cE \quad . \quad . \quad . \quad . \quad (394)$$

where $a = \dfrac{L_1R_2 + L_2R_1}{L_1L_2 - M^2}$, $b = \dfrac{R_1R_2}{L_1L_2 - M^2}$, $c = \dfrac{R_2}{L_1L_2 - M^2}$

of which the solution is (from equations 345 and 348)

$$i = Ae^{m_1t} + Be^{m_2t} + \frac{c}{b}E \quad . \quad . \quad . \quad . \quad (395)$$

where $m_1 = \dfrac{-a + \sqrt{a^2 - 4b}}{2}$ and $m_2 = \dfrac{-a - \sqrt{a^2 - 4b}}{2}$

These expressions are very cumbersome in general terms and actual values will be assumed, namely, $L_1 = L_2 = 0{\cdot}1H$,

$M = 0{\cdot}06H$, $R_1 = R_2 = 4\Omega$, $E = 4V$. Then $a = 125$, $b = 2500$, $c = 625$, $c/b = 0{\cdot}25$. If $E = 4$ then $\frac{c}{b}E = 1$, $\sqrt{a^2 - 4b} = 75$, $m_1 = -25$, $m_2 = -100$ giving

$$i_1 = Ae^{-25t} + Be^{-100t} + 1$$

$$\frac{di_1}{dt} = -25Ae^{-25t} - 100Be^{-100t}$$

and

$$L\frac{di_1}{dt} = -2{\cdot}5Ae^{-25t} - 10Be^{-100t}$$

$$R_1 i_1 = 4Ae^{-25t} + 4Be^{-100t} + 4$$

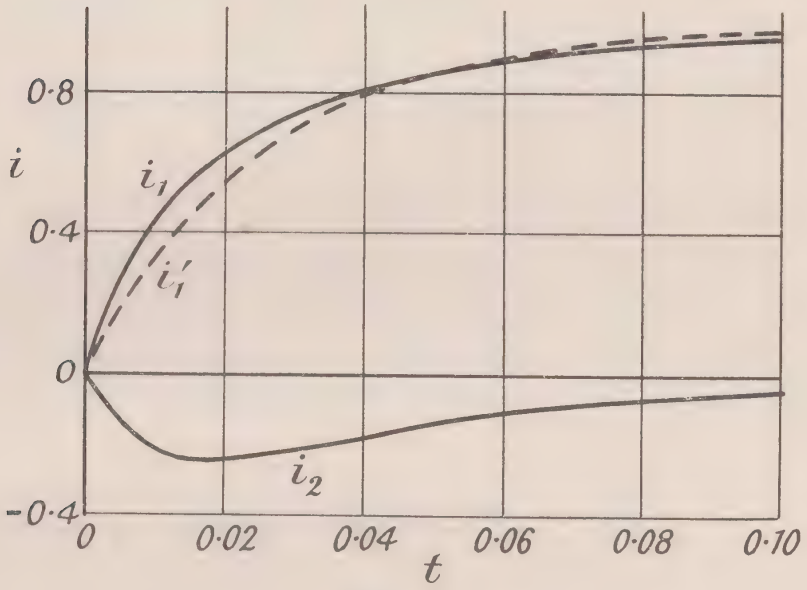


Fig. 82.

Equation 390a becomes

$$4 = L_1\frac{di_1}{dt} + R_1 i_1 + M\frac{di_2}{dt}$$

or

$$M\frac{di_2}{dt} = -1{\cdot}5Ae^{-25t} + 6Be^{-100t}$$

or

$$\frac{di_2}{dt} = -25Ae^{-25t} + 100Be^{-100t}$$

$$L_2\frac{di_2}{dt} = -2{\cdot}5Ae^{-25t} + 10Be^{-100t}$$

$$M\frac{di_1}{dt} = -1{\cdot}5Ae^{-25t} - 6Be^{-100t}$$

But from equation 391*a*

$$L_2\frac{di_2}{dt} + M\frac{di_1}{dt} + R_2 i_2 = 0$$

or
$$R_2 i_2 = 4Ae^{-25t} - 4Be^{-100t}$$
$$i_2 = Ae^{-25t} - Be^{-100t}$$

When $t = 0$, $i_1 = 0$, $i_2 = 0$ hence $A = B = -0{\cdot}5$.

The solution is therefore

$$i_1 = 1 - 0{\cdot}5(e^{-25t} + e^{-100t})$$
$$i_2 = -0{\cdot}5(e^{-25t} - e^{-100t})$$

It should be noted that the general directions of these currents are opposed. Were the secondary open circuited the current in the primary would be $i_1' = 1 - e^{-40t}$.

These three currents are shown plotted in Fig. 82.

The Single Phase Alternator.

The single phase alternator differs from the static transformer in that the mutual inductance between the primary (field system) and the secondary (armature) varies harmonically. The equations defining the primary and secondary currents are

$$e = i_1 R_1 + L_1\frac{di_1}{dt} + \frac{d}{dt}(\mu i_2) \quad . \quad . \quad . \quad (396)$$

$$0 = i_2 R_2 + L_2\frac{di_2}{dt} + \frac{d}{dt}(\mu i_1) \quad . \quad . \quad . \quad (397)$$

where μ is the mutual inductance. Assuming a sinoidal variation as one system rotates with respect to the other one may write $\mu = M \cos \omega t$ where M is the mutual inductance between the two windings when their axes are coincident. Substitution in equations 396 and 397 shows that it is not possible for particular solutions to be of the form

$$i_1 = a_1 \cos \omega t + a_2 \sin \omega t + c$$
$$i_2 = b_1 \cos \omega t + b_2 \sin \omega t$$

The reason for this is clear. If the armature is open circuited ($R_2 = \infty$) an e.m.f. of angular frequency ω will be generated in the armature due to a steady D.C. current in the primary. If a current is taken from the armature then an e.m.f. will be

injected into the primary of magnitude $\frac{d}{dt}(\mu i_2)$. As i_2 contains a component of angular frequency ω, $\mu i_2 (= M \cos \omega t \,.\, i_2)$ possesses a component of angular frequency 2ω and hence from equation 396, i_1 must contain a component of frequency 2ω. It therefore follows that $\mu i_1 (= M \cos \omega t \,.\, i_1)$ contains a component of angular frequency 3ω and so on. Considered from a physical point of view the conditions may be explained thus. Due to rotation of the armature (this condition will be assumed, with stationary field) an e.m.f. of frequency ω is generated in it. This gives rise to a load current of frequency ω which causes an alternating field of frequency ω. This alternating field may be resolved into two rotating fields, one travelling with an angular velocity $+\omega$ with respect to the armature, one with an angular velocity $-\omega$ with respect to the armature. But the armature is rotating with an angular velocity ω with respect to the field, and hence these two components have velocities 2ω and 0 with respect to the field system. The stationary component modifies the strength of the main field. The other component generates a voltage (and current) of frequency 2ω in the field winding. This double frequency current in the field winding gives rise to a field which may be resolved into two rotating components of angular velocity $+2\omega$ and -2ω. As the armature is rotating in these fields it has e.m.f.s generated in it of frequency $+2\omega+\omega$ and $-2\omega+\omega$, frequencies equal to the fundamental and three times the fundamental. Continuing, it is seen that the armature current must therefore contain all odd harmonics and the field current all even harmonics. The general equations may be written as

$$i_1 = I + A_2 \sin 2\omega t + A_4 \sin 4\omega t + A_6 \sin 6\omega t \ldots + B_2 \cos 2\omega t + B_4 \cos 4\omega t + B_6 \cos 6\omega t \quad . \quad . \quad (398)$$

$$i_2 = P_1 \sin \omega t + P_3 \sin 3\omega t + P_5 \sin 5\omega t \ldots + Q_1 \cos \omega t + Q_3 \cos 3\omega t + Q_5 \cos 5\omega t \quad . \quad . \quad (399)$$

Russell (*Alternating Currents*, second edition, Vol. II, pp. 61–4) has considered this problem when R_1 and R_2 are negligible (shortcircuit—with field resistance zero). A more complete solution to this case is given below. Its practical importance may not be very great, but it indicates a method of obtaining a solution to equations of an infinite number of terms.

If $R_1 = R_2 = 0$ equations 396 and 397 may be integrated giving

$$L_1 i_1 + M \cos \omega t \,.\, i_2 = \text{constant}$$

and

$$L_2 i_2 + M \cos \omega t \,.\, i_1 = \text{constant}$$

Substituting from equations 398 and 399 gives

$$L_1\left[I + \sum_{n=1}^{\infty} A_{2n} \sin 2n\omega t + \sum_{n=1}^{\infty} B_{2n} \cos 2n\omega t\right]$$

$$+ M \cos \omega t\left[\sum_{n=1}^{\infty} P_{2n-1} \sin (2n-1)\omega t + \sum_{n=1}^{\infty} Q_{2n-1} \cos (2n-1)\omega t\right]$$

$$= \text{constant}$$

and also

$$L_2\left[\sum_{n=1}^{\infty} P_{2n-1} \sin (2n-1)\omega t + \sum_{n=1}^{\infty} Q_{2n-1} \cos (2n-1)\omega t\right]$$

$$+ M \cos \omega t\left[I + \sum_{n=1}^{\infty} A_{2n} \sin 2n\omega t + \sum_{n=1}^{\infty} B_{2n} \cos 2n\omega t\right]$$

$$= \text{constant}$$

If the left-hand members of these equations be expanded it follows that the coefficient of any sinoidal function must be zero.

Since $\cos \omega t \sin (2n-1)\omega t = \frac{1}{2}[\sin 2n\omega t + \sin (2n-2)\omega t]$ and $\cos \omega t \sin (2n+1)\omega t = \frac{1}{2}[\sin (2n+2)\omega t + \sin 2n\omega t]$ the coefficient of $\sin 2n\omega t$, for any value of n greater than unity, is

$$L_1 A_{2n} + \frac{M}{2}(P_{2n-1} + P_{2n+1})$$

Similarly the coefficient of $\cos 2n\omega t$ is

$$L_1 B_{2n} + \frac{M}{2}(Q_{2n-1} + Q_{2n+1})$$

The coefficient of $\sin (2n+1)\omega t$ is

$$L_2 P_{2n+1} + \frac{M}{2}(A_{2n} + A_{2n+2})$$

and the coefficient of $\cos (2n+1)\omega t$ is

$$L_2 Q_{2n+1} + \frac{M}{2}(B_{2n} + B_{2n+2})$$

Each of these coefficients must be zero.

The initial terms are of somewhat different form. The coefficient of sin $2\omega t$ is

$$L_1A_2 + \frac{M}{2}(P_1 + P_3) = 0 \quad . \quad . \quad . \quad (400)$$

the coefficient of cos $2\omega t$ is

$$L_1B_2 + \frac{M}{2}(Q_1 + Q_3) = 0$$

the coefficient of sin ωt is

$$L_2P_1 + \frac{M}{2}A_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (401)$$

and the coefficient of cos ωt is

$$L_2Q_1 + \frac{M}{2}B_2 + MI = 0$$

In order that all the general coefficients shall be zero it is assumed that there is a solution of the form

$$B_{2n} = \beta B_{2n-2}$$

$$Q_{2n+1} = \alpha Q_{2n-1}$$

where α and β are constants. From the symmetry of the coefficients it is clear that the A's and P's are similarly related, namely

$$A_{2n} = \beta A_{2n-2}$$

and

$$P_{2n+1} = \alpha P_{2n-1}$$

Equations 400 and 401 then become

$$L_1A_2 + \frac{M}{2}(1 + \alpha)P_1 = 0$$

$$L_2P_1 + \frac{M}{2}A_2 = 0$$

These equations can only be simultaneously true if $P_1 = 0$ and $A_2 = 0$. Hence $A_{2n} = 0$ and $P_{2n+1} = 0$. The solution is thus reduced to

$$i_1 = I + B_2 \cos 2\omega t + \mathrm{B}_4 \cos 4\omega t + \mathrm{B}_6 \cos 6\omega t + \ldots$$

$$i_2 = Q_1 \cos \omega t + Q_3 \cos 3\omega t + Q_5 \cos 5\omega t + \ldots$$

$$L_1B_{2n} + \frac{M}{2}(Q_{2n-1} + Q_{2n+1}) = 0$$

$$L_2Q_{2n+1} + \frac{M}{2}(B_{2n} + B_{2n+2}) = 0$$

Substituting in these last two equations one has

(i) $$L_1 B_{2n} + \frac{M}{2} Q_{2n-1}(1+\alpha) = 0$$

$$L_2 Q_{2n-1} \cdot \alpha + \frac{M}{2} B_{2n}(1+\beta) = 0$$

giving $$\alpha L_1 L_2 = \frac{M^2}{4}(1+\alpha)(1+\beta)$$

or (ii) $$L_1 B_{2n-2} \cdot \beta + \frac{M}{2} Q_{2n-1}(1+\alpha) = 0$$

$$L_2 Q_{2n-1} + \frac{M}{2} B_{2n-2}(1+\beta) = 0$$

giving $$\beta L_1 L_2 = \frac{M^2}{4}(1+\alpha)(1+\beta)$$

or $\alpha = \beta = \dfrac{M^2}{4L_1 L_2}(1+\alpha)^2$.

Putting $\dfrac{M^2}{L_1 L_2} = x$ and solving for α gives

$$\alpha = \frac{2}{x} - 1 - \frac{2}{x}\sqrt{1-x} \text{ or}$$

$$\alpha = \frac{x}{4} + \frac{1 \cdot 3}{4 \cdot 6}x^2 + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 10}x^4 \ldots$$

or $$\alpha = \sum_{n=1}^{\infty} \frac{\lfloor 2n-1}{2^{2n-1} \lfloor n-1 \rfloor \lfloor n+1}x^n$$

The coefficient of cos ωt gives

$$L_2 Q_1 + \frac{M}{2} B_2 = -MI$$

and the coefficient of cos $2\omega t$ gives

$$\frac{M}{2}(1+\alpha)Q_1 + L_1 B_2 = 0$$

whence $B_2 = 2\alpha I$, $B_4 = 2\alpha^2 I$, $B_6 = 2\alpha^3 I$, etc.

and $$Q_1 = -\frac{MI}{L_2}(1+\alpha)$$

and i_2

$$= -\frac{MI}{L_2}[(1+\alpha)\cos\omega t + \alpha(1+\alpha)\cos 3\omega t + \alpha^2(1+\alpha)\cos 5\omega t + \ldots]$$

The R.M.S. value of the armature current is

$$I_2 = \frac{MI}{\sqrt{2}L_2}\sqrt{(1+\alpha)^2(1+\alpha^2+\alpha^4+\alpha^6\ \ldots)}$$

$$= \frac{MI}{\sqrt{2}L_2}\sqrt{\frac{1+\alpha}{1-\alpha}} = \frac{MI}{\sqrt{2}L_2}\sqrt{1+2\alpha+2\alpha^2+2\alpha^3\ \ldots}$$

The first few terms are given by

$$I_2 = \frac{MI}{\sqrt{2}L_2}\left[1+\frac{M^2}{2L_1L_2}+\frac{3M^4}{8L_1^2L_2^2}+\frac{5M^6}{16L_1^3L_2^3}+\text{etc.}\right]^{\frac{1}{2}}$$

The values of the various harmonics can be similarly calculated.

Determination of practical values is quite simple if the magnitude of the coupling coefficient of the windings is known. Thus if in a practical case $M = 0{\cdot}5\sqrt{L_1L_2}$ then $x = 0{\cdot}25$ and $\alpha = 0{\cdot}0717$.

Transformer with Capacitative Circuit.

When there is capacitance in transformer circuits the equations are much more complicated than equations 392 and 393. The equations for Fig. 83 are

$$e_1 = i_1R_1 + L_1\frac{di_1}{dt} + \frac{1}{C_1}\int i_1dt + M\frac{di_2}{dt}$$

and

$$e_2 = i_2R_2 + L_2\frac{di_2}{dt} + \frac{1}{C_2}\int i_2dt + M\frac{di_1}{dt}$$

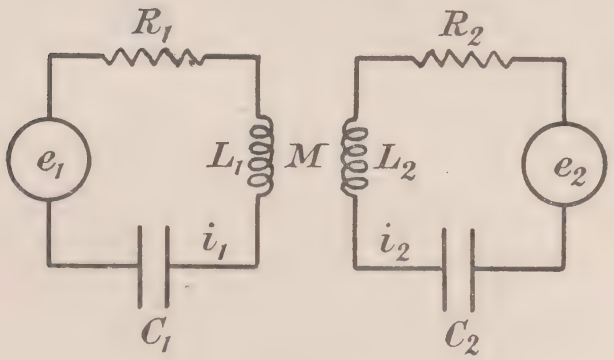


Fig. 83.

On differentiating and writing in operational form these become

$$\left(p^2L_1 + pR_1 + \frac{1}{C_1}\right)i_1 + p^2Mi_2 = pe_1 \quad . \quad . \quad (402)$$

and

$$\left(p^2L_2 + pR_2 + \frac{1}{C_2}\right)i_2 + p^2Mi_1 = pe_2 \quad . \quad . \quad (403)$$

Multiplying equation 402 by $p^2L_2 + pR_2 + \frac{1}{C_2}$ and equation 403 by p^2M and subtracting, one obtains

$$(L_1L_2-M^2)p^4i_1+(R_1L_2+R_2L_1)p^3i_1+\left(\frac{L_1}{C_2}+\frac{L_2}{C_1}+R_1R_2\right)p^2i_1$$

$$+\left(\frac{R_1}{C_2}+\frac{R_2}{C_1}\right)pi_1+\frac{1}{C_1C_2}i_1=L_2p^3e_1-Mp^3e_2+R_2p^2e_1+\frac{1}{C_2}pe_1 \quad (404)$$

and a similar equation in i_2.

This equation generally defies an exact solution though usually a useful approximate solution may be obtained.

Dividing through by $L_1L_2 - M^2$ the equation may be written

$$(p^2 + a_1p + b_1)(p^2 + a_2p + b_2)i_1 = f(e_1, e_2)$$

The transient terms are given by

$$(p^2 + a_1p + b_1)(p^2 + a_2p + b_2)i_1 = 0$$

If the two circuits are identical then the factorisation is simple and the equation may be written as

$$\left(p^2+\frac{R}{L+M}p+\frac{1}{C(L+M)}\right)\left(p^2+\frac{R}{L-M}p+\frac{1}{C(L-M)}\right)i_1=0$$

This equation is satisfied if either

$$\left[p^2 + \frac{R}{L + M}p + \frac{1}{C(L + M)}\right]i_1 = 0$$

or

$$\left[p^2 + \frac{R}{L - M}p + \frac{1}{C(L - M)}\right]i_1 = 0$$

and the solution is

$$i_1=e^{m_1t}(A\cos\omega_1t+B\sin\omega_1t)+e^{m_2t}(A\cos\omega_2t+B\sin\omega_2t)$$

where $$m_1 = -\frac{R}{2(L + M)},\quad m_2 = -\frac{R}{2(L - M)},$$

$$\omega_1 = \sqrt{\frac{1}{C(L + M)} - \frac{R^2}{4(L + M)^2}}$$

and $$\omega_2 = \sqrt{\frac{1}{C(L - M)} - \frac{R^2}{4(L - M)^2}}$$

Two frequencies occur in both the primary and secondary currents. If R is small then

$$\omega_1 \doteqdot \sqrt{\frac{1}{C(L + M)}} \text{ and } \omega_2 \doteqdot \sqrt{\frac{1}{C(L - M)}}$$

The Valve Oscillator.

A thermionic valve is imagined to have the anode and grid potentials adjusted so that it is taking a steady current which varies linearly with either the grid or anode potentials. If changes e_a and e_g are made in the anode and grid potentials respectively the resultant change i_a in the anode current is given by

$$i_a = ae_a + ge_g \quad . \quad . \quad . \quad . \quad . \quad (405)$$

where a is the anode conductance and g the mutual conductance. In the anode lead an oscillatory circuit is connected. Any disturbance of this circuit would cause an oscillatory current to circulate. If the valve were not present this oscillation would naturally die out at a rate depending upon the resistance of the circuit. If, however, a coil coupled to the oscillatory circuit is connected to the grid (Fig. 84), when the valve is in

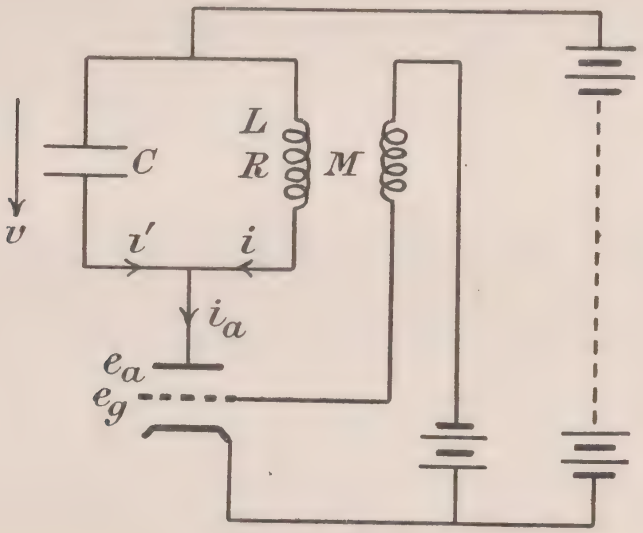


Fig. 84.

operation, the effect is to increase or decrease the effective resistance of the oscillatory circuit according to the direction in which it is connected. In this way the damping of the circuit is increased or decreased. If decreased, and the coupling is made sufficient, the damping may be reduced to zero. In this case an oscillation once started will persist. If the coupling is increased still further the oscillation will grow until limited by the characteristics of the valve. It is proposed to investigate

the steady oscillatory condition. From the figure the following equations are obtained

$$e_a = -v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (406)$$

$$e_g = M\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (407)$$

(the positive value of M is chosen, it is a matter of connection)

$$i_a = i + i' = ae_a + ge_g \quad . \quad . \quad . \quad . \quad (408)$$

$$i' = C\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (409)$$

$$v = iR + L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (410)$$

From equations 409 and 410 one obtains

$$LC\frac{d^2i}{dt^2} + CR\frac{di}{dt} = i'$$

Adding i gives

$$LC\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = ae_a + ge_g$$

$$= -aRi - aL\frac{di}{dt} + gM\frac{di}{dt}$$

or

$$LC\frac{d^2i}{dt^2} + (CR + aL - gM)\frac{di}{dt} + (1 + aR)i = 0 \quad (411)$$

The solution to this equation is oscillatory so long as

$$(CR + aL - gM)^2 < 4LC(1 + aR)$$

This condition will hold in all cases of practical interest when the solution of equation 411 becomes

$$i = Ae^{bt}\sin(\omega t + \phi) \quad . \quad . \quad . \quad . \quad (412)$$

where $$b = \frac{gM - CR - aL}{2LC}$$ and

$$\omega = \sqrt{\frac{1 + aR}{LC} - \frac{(CR + aL - gM)^2}{4LC}}$$

If $gM < CR + aL$ the oscillation will die out; if $gM > CR + aL$ the oscillation will grow; if

$$gM = CR + aL \quad . \quad . \quad . \quad . \quad . \quad (413)$$

the oscillation will be steadily maintained. In this case the angular frequency is

$$\omega = \sqrt{\frac{1 + aR}{LC}} \quad . \quad . \quad . \quad . \quad . \quad (414)$$

In practice R is small and $\omega \doteqdot \sqrt{\frac{1}{LC}}$.

The solution of this problem by vector methods is interesting. It is assumed that a sinoidal solution is possible and an attempt is made to solve the equations. If no solution is obtainable then oscillation is impossible. Thus, assuming that a maintained oscillation of angular frequency ω is possible, equations 406–410 are rewritten as

$$e_a = -v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (406)$$
$$e_g = j\omega Mi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (407a)$$
$$i_a = i' + i = ae_a + ge_g \quad . \quad . \quad . \quad . \quad (408)$$
$$i' = j\omega Cv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (409a)$$
$$v = (R + j\omega L)i \quad . \quad . \quad . \quad . \quad . \quad . \quad (410a)$$

Then from equations 409a and 410a

$$-\omega^2 LCi + j\omega CRi = i'$$

Adding i gives

$$i(1 - \omega^2 LC + j\omega CR) = ae_a + ge_g$$
$$= (-aR - j\omega aL)i + j\omega gMi$$

or $$1 - \omega^2 LC + aR + j\omega CR + j\omega aL - j\omega gM = 0$$

Separating real and imaginary parts gives

$$1 - \omega^2 LC + aR = 0 \quad . \quad . \quad . \quad . \quad (415)$$

and

$$j\omega(CR + aL - gM) = 0 \quad . \quad . \quad . \quad . \quad (416)$$

It is necessary that both these equations be satisfied; it is seen that they can be. Equation 416 gives the condition for maintenance, namely

$$gM = CR + aL \quad . \quad . \quad . \quad . \quad . \quad (414)$$

and equation 415 gives the frequency at which maintenance is possible, namely

$$\omega = \sqrt{\frac{1 + aR}{LC}} \quad . \quad . \quad . \quad . \quad . \quad (415)$$

It will be found that the equations usually break up in this way, the condition for maintenance being given by one component and the frequency by the other.

The Moving Coil Speaker.

The theory of the moving coil speaker closely resembles that of the moving coil galvanometer. It reveals, however, additional features of interest. The circular coil is suspended in a radial field so that a current through it exerts a force in the direction of the axis. If the total length of wire in the coil be l, the intensity of the field H, and i the current in C.G.S. units, then the force P is

$$P = lHi \text{ dynes} = ci \quad . \quad . \quad . \quad . \quad (417)$$

If the coil moves with a velocity u the e.m.f. generated in it (C.G.S. electromagnetic units) is

$$e' = lHu = cu \quad . \quad . \quad . \quad . \quad . \quad (418)$$

Then if m be the mass of the system, ρ the damping coefficient (dynes/cm./sec.) and s the stiffness of the elastic constraint (dynes/cm.), and x the displacement

$$ci = m\frac{d^2x}{dt^2} + \rho\frac{dx}{dt} + sx \quad . \quad . \quad . \quad . \quad (419)$$

If R be the resistance of the coil, L its inductance, and e be the applied e.m.f., then

$$e = L\frac{di}{dt} + Ri + e'$$

or

$$e = L\frac{di}{dt} + Ri + c\frac{dx}{dt} \quad . \quad . \quad . \quad . \quad (420)$$

Written in operational form these equations are

$$(mp^2 + \rho p + s)x = ci \quad . \quad . \quad . \quad . \quad (419a)$$

and

$$(Lp + R)i + cpx = e \quad . \quad . \quad . \quad . \quad (420a)$$

giving

$$[Lmp^3 + (L\rho + Rm)p^2 + (Ls + R\rho + c^2)p + Rs]x = ce \quad (421)$$

and

$$[Lmp^3 + (L\rho + Rm)p^2 + (Ls + R\rho + c^2)p + Rs]i = (mp^2 + \rho p + s)e \quad (422)$$

The solution of these equations is not easy.

The sustained conditions when an alternating e.m.f. is

applied are of importance. Equations 419 and 420 may then be written vectorially (putting $p = j\omega$)

$$(-\omega^2 m + j\omega\rho + s)x = ci \quad . \quad . \quad . \quad (423)$$

and

$$(j\omega L + R)i + j\omega cx = e \quad . \quad . \quad . \quad . \quad (424)$$

Equation 423 is of interest. It may be written

$$ci = \left[\rho + j\left(\omega m - \frac{s}{\omega}\right)\right] . j\omega x \quad . \quad . \quad . \quad (425)$$

ci is the force, $j\omega x$ the vector velocity and therefore

$$\rho + j\left(\omega m - \frac{s}{\omega}\right)$$

may be termed the vector mechanical impedance. The value of this mechanical impedance is

$$Z_m = \sqrt{\rho^2 + \left(\omega m - \frac{s}{\omega}\right)^2} \quad . \quad . \quad . \quad . \quad (426)$$

The mechanical power factor $\cos\theta_m = \rho/Z_m$. Substituting from equation 423 in equation 424 gives

$$e = i\left[R + j\omega L + \frac{c^2}{\rho + j\left(\omega m - \frac{s}{\omega}\right)}\right] \quad . \quad . \quad (427)$$

This result should be noted. Due to motion the equivalent vector impedance of the coil is increased by

$$\frac{c^2}{\rho + j\left(\omega m - \frac{s}{\omega}\right)}$$

The denominator is the vector mechanical impedance. This increase in impedance can be represented by a resistance R_m in series with a capacitance C_m, for it may be written

$$\frac{c^2}{\rho + j\left(\omega m - \frac{s}{\omega}\right)} = \frac{c^2}{Z^2{}_m}\left[\rho - j\left(\omega m - \frac{s}{\omega}\right)\right]$$

$$= \frac{c^2\rho}{Z_m^2} + \frac{c^2\left(\omega m - \frac{s}{\omega}\right)}{jZ^2{}_m}$$

$$= \frac{c^2\rho}{Z_m^2} + \frac{1}{j\omega C_m}$$

$$C_m = \frac{Z^2{}_m}{\omega c^2\left(\omega m - \frac{s}{\omega}\right)} = \frac{Z^2{}_m\left(m - \frac{s}{\omega^2}\right)}{c^2\left(\omega m - \frac{s}{\omega}\right)^2}$$

or

$$R_m = \frac{c^2\rho}{Z^2{}_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (428)$$

$$C_m = \frac{m - \frac{s}{\omega^2}}{c^2 \sin^2 \theta_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (429)$$

Both R_m and C_m vary with the frequency.

To examine the effect of the motion a simple case will be evaluated. The coil will be supposed to be unloaded (not fixed to a diaphragm) and to be freely suspended. In this case $\rho = 0$ and $s = 0$ giving $R_m = 0$ and $C_m = \frac{m}{c^2}$. Equation 427 then becomes

$$e = i\left[R + j\omega L + \frac{c^2}{j\omega m}\right] . \quad . \quad . \quad . \quad (430)$$

Equation 423 similarly simplifies to

$$-\omega^2 mx = ci \quad . \quad . \quad . \quad . \quad . \quad . \quad (431)$$

The following constants will be assumed: $m = 5$ gm.; $L = 0{\cdot}15$ henry; $R = 1000$ ohms; coil of 1000 turns, diameter 5 cm.; $H = 10{,}000$ lines/sq. cm.; applied voltage 100 R.M.S. at $\omega = 500$ giving $\omega L = 75\Omega$.

If the coil were locked the equation would be

$$100 = i(1000 + 75j)$$

The current would be 0·0997, lagging by a small angle behind the voltage. The voltage on inductance would be 7·5 and on resistance 99·7.

The coil being allowed to move, a considerable voltage is generated in it due to its motion. This voltage appears in the equation as the voltage upon C_m, the coil behaves as a condenser, there being resistance and inductance in the leads. For the coil

$$l = 5000\pi \text{ and } lH = 5\pi \times 10^7 = c$$

$$C_m = \frac{m}{c^2} = \frac{5}{25\pi^2 \times 10^{14}} = 0{\cdot}2026 \times 10^{-15}$$

$$= 0{\cdot}2026\ \mu F$$

At $\omega = 500$ the vector impedance of this condenser is $-j9870$ ohms and from equation 430

$$e = i[1000 + j75 - j9870] = i[1000 - j9795]$$

giving $i = 0{\cdot}00103 + j0{\cdot}0101$ amperes.

The current is only just over one-tenth of its value when the coil was locked and has advanced in phase by nearly 90°. The magnitude of the motion is given by equation 431.

In absolute units $i = (0{\cdot}103 + j1{\cdot}01) \times 10^{-3}$ giving

$$x = \frac{ci}{-\omega^2 m} = -\frac{5\pi \times 10^4(0{\cdot}103 + j1{\cdot}01)}{25 \times 10^4 \times 5}$$

$$= -\frac{\pi}{25}(0{\cdot}103 + j1{\cdot}01) \text{ cm.}$$

The actual magnitude of this is 0·1275 cm. R.M.S. The total swing is therefore $2\sqrt{2} \times 0{\cdot}1275 = 0{\cdot}361$ cm. The R.M.S. velocity of the coil is

$$j\omega x = 20\pi(1{\cdot}01 - j0{\cdot}103)$$

and the e.m.f. generated in it is c times this or

$$10^9\pi^2(1{\cdot}01 - j0{\cdot}103) \text{ or}$$
$$10\pi^2(1{\cdot}01 - j0{\cdot}103) \text{ volts}$$
$$= 99{\cdot}7 - j10{\cdot}2 \text{ volts.}$$

The same result is naturally obtained by considering the equivalent capacitance. The voltage on this is

$$\frac{1}{j\omega C_m}(0{\cdot}103 + j1{\cdot}01) \times 10^{-2}$$

$$\frac{1}{C_m} = \frac{10^6\pi^2}{2} \text{ and } \frac{1}{\omega C_m} = 10^3\pi^2$$

giving the voltage as $10\pi^2(1{\cdot}01 - j0{\cdot}103)$ as before.

The voltage on resistance is $1{\cdot}0 + j10{\cdot}1$ and upon inductance $-0{\cdot}7_5 + j0{\cdot}1$. It will be seen that within the limits of accuracy of the arithmetic these three voltages add up to the applied voltage of 100. The motional e.m.f. is by far the greatest and approximates to the applied voltage. The R.M.S. values of the voltages are: applied, 100; motional, 100·2; resistive, 10·2; inductive, 0·8. The capacitative effect of the

moving coil is clearly indicated when $\omega = 5736$. Then $\omega L = \dfrac{1}{\omega C_m} = 860{\cdot}4$. The current is then 0·1 amp, the motional voltage 86, the inductive voltage 86, and the resistive voltage 100.

The Telephone Diaphragm.

The following example is quoted:

"That part of the impedance of a telephone receiver which is due to the motion of the diaphragm is given by the expression $A^2/\left\{r + j\left(\omega m - \dfrac{s}{\omega}\right)\right\}$, where $A = 2BN/R$, and B = flux density in the air gap, N = number of turns of receiver coils, R = reluctance of the circuit to alternating fluxes; r, m and s are the equivalent mechanical resistance, mass and stiffness of the receiver diaphragm. Find the dimensions of each quantity and check that the expression has the dimensions of a resistance." (Lond. Univ. B.Sc. Eng. Ext., 1934, Electrical Measurements and Measuring Instruments, I. 2.)

It is not proposed to consider the actual question but instead the expression quoted for the motional impedance.

This is deduced under the assumption that any change of flux due to movement of the diaphragm in the magnetic field is negligible and that the whole change of flux is due to the current flowing through the receiver coils.

If the steady flux due to the magnets is B and the superposed flux due to current in the coils is b then the pull on the diaphragm per square centimetre is $\dfrac{(B+b)^2}{8\pi}$. The extra pull due to b is $\dfrac{2Bb + b^2}{8\pi}$ dynes/sq. cm. As b is small compared with B this may be said to be $\dfrac{Bb}{4\pi}$. If the area of one of the poles is S the total extra pull due to b is $\dfrac{BbS}{2\pi}$. If i is the current in C.G.S. units the extra flux per pole is $bS = \dfrac{4\pi Ni}{R}$ so that the force P is $\dfrac{2BN}{R}i = Ai$.

The equation of motion is therefore

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + sx = P = Ai \quad . \quad . \quad . \quad (432)$$

Comparing this with equation 419 and its solution under sinoidal conditions (equation 427) one obtains

$$Z_m = \frac{A^2}{r + j\left(\omega m - \frac{s}{\omega}\right)}$$

CHAPTER XV

DIFFERENTIAL EQUATIONS (*continued*)

The Homogeneous Linear Differential Equation.

The homogeneous linear differential equation is of the form

$$x^n\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} \ldots P_{n-1}\,x\frac{dy}{dx} + P_ny = X \quad (433)$$

where $P_1, P_2 \ldots P_n$ are constants and X is a function of x. The equation can be transformed into a linear differential equation with constant coefficients by the substitution

$$x = e^z \text{ or } z = \log x$$

for $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \text{ or } x\frac{dy}{dx} = \frac{dy}{dz}$$

$x\dfrac{d}{dx}$ is therefore established as a linear operator of the same nature as D, the dependent variable being z or $\log x$.

Writing $x\dfrac{d}{dx}$ or $\dfrac{d}{dz}$ as θ then

$$\theta \,.\, S = x \,.\, D \,.\, S$$

or

$$\theta \equiv x \,.\, D \quad \ldots\ldots\ldots \quad (434)$$

$$\begin{aligned}\theta^2 \,.\, S &= xDxDS \\ &= x(xD^2S + DS) \\ &= x^2D^2S + \theta S\end{aligned}$$

$$x^2D^2 = \theta^2 - \theta = \theta(\theta - 1) \quad \ldots \quad (435)$$

$$\begin{aligned}\theta^3 \,.\, S &= xD(x^2D^2S + xDS) \\ &= x^3D^3S + 2x^2D^2S + x^2D^2S + xDS \\ &= x^3D^3S + 3\theta(\theta - 1)S + \theta \,.\, S\end{aligned}$$

$$\begin{aligned}x^3D^3 &= \theta^3 - 3\theta^2 + 2\theta \\ &= \theta(\theta - 1)(\theta - 2) \quad \ldots\ldots\ldots \quad (436)\end{aligned}$$

and generally

$$x^nD^n = \theta(\theta - 1)(\theta - 2) \ldots (\theta - n + 1) \quad (437)$$

Equation 433 then becomes
$[\theta(\theta-1)\ldots(\theta-n+1)+P_1\theta(\theta-1)\ldots(\theta-n+2)\ldots+P_n]y=Z$
where Z is the function of z into which X changes.

Example.

Solve $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=x^m$.

Writing $x=e^z$, $x^m=e^{mz}$, and the equation becomes

$$[\theta(\theta-1)-3\theta+4]y=e^{mz}$$
$$(\theta^2-4\theta+4)y=e^{mz}$$
$$y=e^{2z}(A+Bz)+\frac{e^{mz}}{m^2-4m+4}$$

or
$$y=x^2(A+B\log x)+\frac{x^m}{(m-2)^2}$$

Certain equations not of the homogeneous linear form may be transformed by a suitable substitution. For instance if the terms in the equation are of the form $(a+bx)^r\frac{d^ry}{dx^r}$ one may write

$$w=a+bx$$

then

$$(a+bx)^r\frac{d^ry}{dx^r}=w^rb^r\frac{d^ry}{dw^r} \quad . \quad . \quad . \quad . \quad (438)$$

Example.

Solve $(x+a)^2\frac{d^2y}{dx^2}-4(x+a)\frac{dy}{dx}+6y=x$

$$w^2\frac{d^2y}{dx^2}-4w\frac{dy}{dx}+6y=w-a$$

put $w=e^z$

$$[\theta(\theta-1)-4\theta+6]y=e^z-a$$
$$[\theta^2-5\theta+6]y=e^z-a$$
$$y=Ae^{3z}+Be^{2z}+\frac{e^z}{2}-\frac{a}{6}$$
$$=A(x+a)^3+B(x+a)^2+\frac{3x+2a}{6}$$

Exact Linear Differential Equations.

When an equation could have been derived by differentiation of another of lower order it is said to be exact. The equation may be of the form

$$X_0D^ny + X_1D^{n-1}y + X_2D^{n-2}y \ldots X_ny = X \quad . \quad (439)$$

This is directly integrable once if the left-hand member can be integrated once. The equation to be considered is therefore

$$X_0D^ny + X_1D^{n-1}y + X_2D^{n-2}y \ldots X_ny = 0 \quad . \quad (440)$$

This is

$$D(X_0D^{n-1}y) + (X_1 - DX_0)D^{n-1}y + X_2D^{n-2}y \ldots X_ny = 0$$

or

$$D[X_0D^{n-1}y + (X_1 - DX_0)D^{n-2}y] + (X_2 - DX_1 + D^2X_0)D^{n-2}y \ldots X_ny = 0$$

Continuing, one finally obtains

$$D[X_0D^{n-1}y + (X_1 - DX_0)D^{n-2}y \ldots \{X_{n-1} - DX_{n-2} \ldots (-1)^{n-1}D^{n-1}X_0\}y] + (X_n - DX_{n-1} + D^2X_{n-2} \ldots (-1)^nD^nX_0)y = 0$$

The equation is exact if the coefficient of y is zero, namely if

$$X_n - DX_{n-1} + D^2X_{n-2} \ldots (-1)^nD^nX_0 = 0 \quad . \quad (441)$$

and then the first integral of the equation is

$$X_0D^{n-1}y + (X_1 - DX_0)D^{n-2}y \ldots [X_{n-1} - DX_{n-2} \ldots (-1)^{n-1}D^{n-1}X_0]y = A \quad . \quad (442)$$

Example.

Solve $x\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + 2y = 0$

$$X_2 = 2 \quad X_1 = 2x \quad X_0 = x$$
$$X_2 = 2 \quad DX_1 = 2 \quad D^2X_0 = 0$$
$$X_2 - DX_1 + D^2X_0 = 0.$$

Hence the equation is exact.

The first integral is

$$x\frac{dy}{dx} + (2x - 1)y = A$$

This equation is not exact but it is a linear differential equation of the first order and may be written

$$\frac{dy}{dx} + \left(\frac{2x - 1}{x}\right)y = \frac{A}{x}$$

The integrating factor is $e^{\int 2-\frac{1}{x}dx} = e^{2x-\log x} = \frac{e^{2x}}{x}$

Then
$$\frac{e^{2x}}{x}\frac{dy}{dx} + \frac{e^{2x}}{x}\left(2 - \frac{1}{x}\right)y = \frac{Ae^{2x}}{x^2}$$

$$y\frac{e^{2x}}{x} = \int Ae^{2x}x^{-2}dx + B$$

or
$$y = xe^{-2x}\left[\int Ae^{2x}x^{-2}dx + B\right]$$

Sometimes an equation which is not exact may be made so by use of an integrating factor, for instance

$$x^2\frac{d^2y}{dx^2} + (3x^2 + 2x)\frac{dy}{dx} + 3xy = 0$$

$$X_2 = 3x,\ DX_1 = 6x + 2,\ D^2X_0 = 2$$
$$X_2 - DX_1 + D^2X_0 = -3x$$

and so the equation is not exact.

Suppose it be multiplied by x^m, then it becomes

$$x^{m+2}\frac{d^2y}{dx^2} + (3x^{m+2} + 2x^{m+1})\frac{dy}{dx} + 3x^{m+1}y = 0$$

Now
$$X_2 = 3x^{m+1}$$
$$-DX_1 = (-2m - 2)x^m - (3m + 6)x^{m+1}$$
$$D^2X_0 = (m + 2)(m + 1)x^m$$

The sum of these can be zero if

$$3 - 3m - 6 = 0$$

and
$$-2m - 2 + m^2 + 3m + 2 = 0$$

Both these equations can be satisfied if $m = -1$ when the equation becomes

$$x\frac{d^2y}{dx^2} + (3x + 2)\frac{dy}{dx} + 3y = 0$$

The first integral of this is

$$x\frac{dy}{dx} + (3x + 1)y = 0$$

It is now reduced to a linear equation of the first order and can be directly solved.

Sometimes an equation can be solved by trial, working on the lines of the general method suggested for the exact linear

differential equation. For instance the equation

$$x^2y\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} + 2y\right)^2 - 3y^2 = 0$$

may be considered. This is rewritten as

$$x^2y\frac{d^2y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^2 + 4xy\frac{dy}{dx} + y^2 = 0$$

The first term appears to be derived from the differentiation of $x^2y\frac{dy}{dx}$. This is tried

$$\frac{d}{dx}\left(x^2y\frac{dy}{dx}\right) = x^2y\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2$$

Subtracting this from the left-hand side of the equation leaves $2xy\frac{dy}{dx} + y^2$, but

$$\frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}$$

and so the original equation becomes

$$\frac{d}{dx}\left(xy^2 + x^2y\frac{dy}{dx}\right) = 0$$

which on integration gives

$$xy^2 + x^2y\frac{dy}{dx} = A$$

$x^2y\frac{dy}{dx}$ appears to be derived from differentiating $\frac{1}{2}x^2y^2$. Trying this, one obtains

$$\frac{d}{dx}\left(\frac{1}{2}x^2y^2\right) = xy^2 + x^2y\frac{dy}{dx}$$

And so on integration the equation gives

$$\tfrac{1}{2}x^2y^2 = Ax + B \quad \text{or}$$
$$x^2y^2 = A'x + B'$$

Factorisation of the Operator.

In some cases the left-hand member of the equation may be factorisable. For instance

$$x\frac{d^2y}{dx^2} + (x - 1)\frac{dy}{dx} - y = x^2 \quad . \quad . \quad . \quad (443)$$

may be written

$$(xD - 1)(D + 1)y = x^2 \quad . \quad . \quad . \quad . \quad (444)$$

If $(D + 1)y$ is put equal to v the equation becomes

$$(xD - 1)v = x^2$$

or

$$\frac{dv}{dx} - \frac{1}{x}v = x$$

This is a linear equation of the first degree. The integrating factor is

$$e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

The equation is then

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = 1$$

whence $\frac{v}{x} = x + a$ or $v = x^2 + ax$

Then $(D + 1)y = x^2 + ax$

Solving this as a linear differential equation with constant coefficients the complementary function is $y = be^{-x}$. The particular integral is given by

$$y = \frac{1}{D+1}(x^2 + ax) = (1 - D + D^2 \ldots)(x^2 + ax)$$

$$= x^2 + ax - 2x - a + 2 = x^2 + (a-2)(x-1) = x^2 + c(x-1)$$

The complete solution is $y = be^{-x} + x^2 + c(x - 1)$.

A warning is necessary here. D is a commutative operator and factorisation of expressions in D by algebraical methods is therefore permissible; this has already been shown. But $\theta(\equiv xD)$ is not commutative with D. Clearly an expression such as $xD - 1$ may precede D and the rules of algebra be applicable, but it cannot succeed it under the same conditions. If there be any doubt the supposed identity should be checked, remembering that expressions such as $D + 1$ and $xD - 1$ are operators.

Equation 443 may be written in the form 444 but it is not permissible to interchange $xD - 1$ and $D + 1$. This may be checked

$$(D + 1)y = \frac{dy}{dx} + y$$

$$D(D + 1)y = \frac{d^2y}{dx^2} + \frac{dy}{dx}$$

$$xD(D + 1)y = x\frac{d^2y}{dx^2} + x\frac{dy}{dx}$$

$$\therefore\ (xD-1)(D+1)y = x\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{dy}{dx} - y$$

or
$$(xD-1)(D+1)y = x\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} - y$$

Again
$$(xD-1)y = x\frac{dy}{dx} - y$$

$$D(xD-1)y = \frac{dy}{dx} + x\frac{d^2y}{dx^2} - \frac{dy}{dx} = x\frac{d^2y}{dx^2}$$

$$(D+1)(xD-1)y = x\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$$

Hence $(xD-1)(D+1) \neq (D+1)(xD-1)$.

This has already been seen when considering the operator θ, but the repetition is permissible.

Equation in Commutative Operators.

If A, B, C, etc., represent a number of commutative operators occurring in the equation

$$A \,.\, B \,.\, C \,.\, y = 0 \quad . \quad . \quad . \quad . \quad . \quad (445)$$

then $C \,.\, y = 0$ is a solution. Since the operators are commutative $C \,.\, A \,.\, B \,.\, y = 0$ and similarly $B \,.\, y = 0$ is a solution and similarly $A \,.\, y = 0$ is a solution. Further, the sum of the solutions of these separate equations is a solution of equation 445. This principle has already been employed in the solution of the linear differential equation with constant coefficients (see equations 275 and 276, p. 155). Many practical problems give rise to equations in which this principle may be invoked and a great simplification is effected thereby. A knowledge of the physical conditions may assist in effecting the operational factorisation. Much of this is already possessed in the case of a practical problem. For instance, an equation which occurred in a study of the vibration of loud-speaker diaphragms was

$$\frac{d^4y}{dx^4} + \frac{2}{x}\frac{d^3y}{dx^3} - \frac{1}{x^2}\frac{d^2y}{dx^2} + \frac{1}{x^3}\frac{dy}{dx} - k^4y = 0 \text{ *} \quad . \quad (446)$$

This equation may be written

$$(\Phi^2 + k^2)(\Phi^2 - k^2)y = 0 \quad . \quad . \quad . \quad (447)$$

* *Phil. Mag.*, May 1930, p. 889.

where $$\Phi^2 \equiv \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$$

Now the solution of

$$(\Phi^2 - k^2)y = 0 \quad . \quad . \quad . \quad . \quad . \quad (448)$$

is clearly a solution of equation 447 and since $\Phi^2 + k^2$ and $\Phi^2 - k^2$ can be readily shown to be commutative the solution of

$$(\Phi^2 + k^2)y = 0 \quad . \quad . \quad . \quad . \quad . \quad (449)$$

is also a solution of equation 447. The sum of the two solutions of equations 448 and 449 is therefore a solution of equation 447.

Equations 448 and 449 are Bessel equations (Chap. XVII). The solution was expected to involve Bessel functions and therefore factorisation of equation 446 was tried in the form of 447 and found to be possible. It was definitely a case of knowledge of the physical conditions suggesting an analytical method.

Equations which do not contain *y* Directly.

If an equation does not contain y directly then one may write $\frac{dy}{dx} = q$, $\frac{d^2y}{dx^2} = \frac{dq}{dx}$, $\frac{d^ny}{dx^n} = \frac{d^{n-1}q}{dx^{n-1}}$. An equation is then obtained in q instead of y but one lower in order than the original equation.

Equations of the Second Order.

A large number of physical problems give rise to equations of the second order and the method of solving these will be considered. Equations of the form

$$\frac{d^2y}{dx^2} = f(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (450)$$

may be solved by multiplying both sides by $2\frac{dy}{dx}$ giving

$$2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 2\frac{dy}{dx} f(y)$$

then $$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)dy + c$$

For instance if

$$\frac{d^2y}{dx^2} = a^2y$$

$$\left(\frac{dy}{dx}\right)^2 = a^2y^2 + \text{constant}$$

$$\left(\frac{dy}{dx}\right)^2 = a^2y^2 + a^2c^2$$

$$\frac{dy}{dx} = a\sqrt{y^2 + c^2}\,; \quad \frac{dy}{\sqrt{y^2 + c^2}} = a\,dx$$

$$\sinh^{-1}\frac{y}{c} = ax + b$$

or
$$y = c \sinh (ax + b) \quad . \quad . \quad . \quad . \quad (451)$$

Complete Solution when an Integral is Known.

The general linear equation of the second degree may be written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \quad . \quad . \quad . \quad . \quad (452)$$

where P, Q and X are functions of x. If one solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad . \quad . \quad . \quad . \quad (453)$$

be known then the equation may be reduced to one of the first order. If $y = y_1$ be the known solution of equation 453 then it is assumed that $y = y_1v$ is the complete solution

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1\frac{dv}{dx}$$

and
$$\frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + 2\frac{dy_1}{dx}\cdot\frac{dv}{dx} + y_1\frac{d^2v}{dx^2}$$

Substituting these values in equation 452 gives

$$y_1\frac{d^2v}{dx^2} + 2\frac{dy_1}{dx}\cdot\frac{dv}{dx} + v\frac{d^2y_1}{dx^2} + vP\frac{dy_1}{dx} + vQy_1 + Py_1\frac{dv}{dx} = X \quad . \quad . \quad (454)$$

But since y_1 is a solution of equation 453 the sum of the 3rd, 4th and 5th terms of the left-hand member of this equation is zero and hence

$$y_1\frac{d^2v}{dx^2} + \left(Py_1 + 2\frac{dy_1}{dx}\right)\frac{dv}{dx} = X$$

or
$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx} = \frac{X}{y_1} \quad . \quad . \quad . \quad (455)$$

This equation does not contain v directly and so, writing q for $\frac{dv}{dx}$, it becomes

$$\frac{dq}{dx} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)q = \frac{X}{y_1}$$

This is a linear differential equation of the first order in q. To find the integrating factor we must first determine

$$\int\left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)dx$$

$$\int\left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)dx = \int P\,dx + \log {y_1}^2$$

The integrating factor is therefore ${y_1}^2 e^{\int P\,dx}$ and therefore

$$q{y_1}^2 e^{\int P\,dx} = \int y_1 X e^{\int P\,dx}dx + A$$

$$\text{or } q = \frac{dv}{dx} = A{y_1}^{-2}e^{-\int P\,dx} + {y_1}^{-2}e^{-\int P\,dx}\int y_1 X e^{\int P\,dx}dx$$

On integrating this and putting $y = y_1 v$ this gives

$$y = By_1 + Ay_1\int {y_1}^{-2}e^{-\int P\,dx}dx$$

$$+ y_1\int\left[{y_1}^{-2}e^{-\int P\,dx}\int y_1 X e^{\int P\,dx}dx\right]dx \quad . \quad . \quad (456)$$

The original solution y_1 is a factor of this and this solution contains two arbitrary constants. It is therefore the complete solution.

Example.

$$\text{Solve } (3 - x)\frac{d^2y}{dx^2} - (9 - 4x)\frac{dy}{dx} + (6 - 3x)y = 0.$$

An examination of this equation shows that the sum of the three coefficients is zero. It therefore follows that a solution is $y = \varepsilon^x$; this can be readily checked. Putting $y = ve^x$, equation 456 might then be used as a formula but it is hardly likely to be remembered and it is best to make the actual substitution

$$\frac{dy}{dx} = ve^x + e^x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = ve^x + 2e^x\frac{dv}{dx} + e^x\frac{d^2v}{dx^2}$$

Then

$$(3-x)\left(ve^x + 2e^x\frac{dv}{dx} + e^x\frac{d^2v}{dx^2}\right) - (9-4x)\left(ve^x + e^x\frac{dv}{dx}\right) + (6-3x)ve^x = 0$$

$$(3-x)\left(\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - (9-4x)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} - \left(\frac{3-2x}{3-x}\right)\frac{dv}{dx} = 0$$

Putting $\frac{dv}{dx} = q$ this becomes

$$\frac{dq}{dx} - \left(2 - \frac{3}{3-x}\right)q = 0$$

The solution of this is

$$q = Ae^{\int\left(2-\frac{3}{3-x}\right)dx}$$

or

$$q = A(3-x)^3e^{2x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = A(27 - 27x + 9x^2 - x^3)e^{2x}$$

$$\int e^{2x}dx = \frac{e^{2x}}{2}\,; \quad \int xe^{2x}dx = e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right)$$

$$\int x^2e^{2x}dx = e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)$$

$$\int x^3e^{2x}dx = e^{2x}\left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}\right)$$

whence

$$v = B + \frac{A}{8}e^{2x}(183 - 150x + 42x^2 - 4x^3)$$

or

$$y = Be^x + A'e^{3x}(183 - 150x + 42x^2 - 4x^3)$$

Change of the Dependent Variable.

When a solution corresponding to the complementary function is not known it may still be possible to transform the equation to an integrable type by change of the dependent variable. Thus in equation 452 one may write $y = y_1v$ where y_1 is some function of x and then one obtains, as in the previous section, equation 454, which may be written

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx} + \frac{1}{y_1}\left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\right)v = \frac{X}{y_1} \quad . \quad . \quad (457)$$

Now the coefficient of $\frac{dv}{dx}$ or of v can be given any value by a proper choice of y_1. If the coefficient of v be made zero then the equation becomes one not containing v directly and is identical with the form solved in the previous section. It may however be convenient to make the coefficient of $\frac{dv}{dx}$ zero. If this be done then

$$P + \frac{2}{y_1}\frac{dy_1}{dx} = 0$$

and

$$y_1 = e^{-\frac{1}{2}\int P\,dx} \quad . \quad . \quad . \quad . \quad . \quad (458)$$

$$\frac{dy_1}{dx} = -\frac{P}{2}y_1\,; \quad \frac{d^2y_1}{dx^2} = \left(\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}\right)y_1$$

and so the coefficient of v becomes $Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$. Equation 457 then becomes

$$\frac{d^2v}{dx^2} + \left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right)v = Xe^{\frac{1}{2}\int P\,dx} \quad . \quad . \quad (459)$$

Example.

Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = 0$

$$y = y_1v\,; \quad P = -2\tan x\,; \quad Q = 5$$

$$y_1 = e^{-\frac{1}{2}\int P\,dx} = e^{\int \tan x\,dx} = \sec x\,;$$

$$\frac{dP}{dx} = -2\sec^2 x\,; \quad P^2 = 4\tan^2 x$$

$$Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 5 + \sec^2 x - \tan^2 x = 6$$

and the equation becomes

$$\frac{d^2v}{dx^2} + 6v = 0$$

$$v = A\cos\sqrt{6}x + B\sin\sqrt{6}x$$

$$y = \sec x(A\cos\sqrt{6}x + B\sin\sqrt{6}x)$$

CHAPTER XVI

THE SOLUTION OF DIFFERENTIAL EQUATIONS IN SERIES

In the immediately preceding chapters methods have been given for solving differential equations of the form $D^2y+PDy+Qy=0$ where P and Q were functions of x. The solutions, which involved two arbitrary constants, were in the form of algebraic or trigonometrical functions which, in a large number of cases, were infinite series, for example e^x, $\cos x$, $\log x$, etc., the forms e^x, $\cos x$, $\log x$, etc., being shorthand symbols for the series. A solution in terms of an infinite series is always possible, though often not possible in a series which has an accepted shorthand symbol. In equations of important form, recurring in practical problems, the series obtained may have a symbol allotted and the function be defined in terms of the equation of which it is a solution.

The method adopted is to assume that there is a solution of the form

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} \quad . \quad . \quad . \quad (460)$$

A simple example illustrates. Suppose the equation to be

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \quad . \quad . \quad . \quad . \quad (461)$$

Then if

$$y = a_0x^m + a_1x^{m+1} + \ldots$$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

$$2x\frac{d^2y}{dx^2} = 2m(m-1)a_0x^{m-1} + 2(m+1)ma_1x^m + 2(m+2)(m+1)a_2x^{m+1} + \ldots$$

And so in $2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y$ the coefficients of the various powers of x are

$$x^{m-1}\ ;\ [2m(m-1)+m]a_0 = 2m(m-\tfrac{1}{2})a_0 \quad . \quad . \quad . \quad . \quad . \quad (462)$$

x^m ; $[2(m+1)m+m+1]a_1+a_0=2(m+1)(m+\frac{1}{2})a_1+a_0$. (463)

x^{m+1} ; $2(m+2)(m+1\frac{1}{2})a_2+a_1$ . . . . . (464)

x^{m+2} ; $2(m+3)(m+2\frac{1}{2})a_3+a_2$ . . . . . (465)

x^{m+r} ; $2(m+r+1)(m+r+\frac{1}{2})a_{r+1}+a_r$ . . . . . (466)

All these coefficients must be zero. Equating to zero the coefficient of the lowest power of x one obtains what is commonly called the indicial equation. In this case the indicial equation is $m(m-\frac{1}{2})=0$ giving two possible values for m namely 0 and $\frac{1}{2}$. These values give two possible solutions and, as a_0 is arbitrary and can be made different in the two solutions, a complete solution is usually possible. Taking first $m=0$ then equation 463 gives $a_1=-a_0$; equation 464 gives $6a_2=-a_1$, or $a_2=\dfrac{a_0}{6}$.

Considering the equations generally, equation 466 gives

$$a_{r+1}=-\frac{1}{(r+1)(2r+1)}a_r$$

giving $a_1=-a_0$; $a_2=\dfrac{1}{2\,.\,3}a_0$; $a_3=-\dfrac{1}{2\,.\,3\,.\,3\,.\,5}a_0$

$$a_4=\frac{1}{2\,.\,3\,.\,4\,.\,3\,.\,5\,.\,7}a_0, \text{ etc.}$$

and the first solution is

$$y_1=a_0\left[1-x+\frac{x^2}{2\,.\,3}-\frac{x^3}{(2\,.\,3)(3\,.\,5)}+\frac{x^4}{(2\,.\,3\,.\,4)(3\,.\,5\,.\,7)}-\ldots\right] \quad (467)$$

Taking the value $m=\frac{1}{2}$ gives

$$a_{r+1}=-\frac{1}{(r+1)(2r+3)}a_r$$

$$a_1=-\frac{1}{1\,.\,3}a_0;\ a_2=\frac{1}{(1\,.\,2)(3\,.\,5)}a_0;\ a_3=-\frac{1}{(1\,.\,2\,.\,3)(3\,.\,5\,.\,7)}a_0$$

etc.

giving as the second solution (writing b_0 instead of a_0)

$$y_2=b_0\left[x^{\frac{1}{2}}-\frac{x^{3/2}}{1\,.\,3}+\frac{x^{5/2}}{(1\,.\,2)(3\,.\,5)}-\frac{x^{7/2}}{(1\,.\,2\,.\,3)(3\,.\,5\,.\,7)}+\ldots\right] \quad (468)$$

The complete solution is $y=y_1+y_2$.

Values of m differing by an Integer.

Substituting equation 460 in

$$\frac{d^2y}{dx^2} - y = 0 \quad . \quad . \quad . \quad . \quad . \quad (469)$$

gives as the coefficients of the various powers of x

$$\begin{aligned} x^{m-2} &; \ m(m-1)a_0 \\ x^{m-1} &; \ (m+1)m\,a_1 \\ x &; \ (m+2)(m+1)a_2 - a_0 \\ x^{m+1} &; \ (m+3)(m+2)a_3 - a_1 \\ x^{m+r} &; \ (m+r+2)(m+r+1)a_{r+2} - a_r \end{aligned}$$

The indicial equation gives $m = 0$ or 1. But if $m = 0$ the second equation is satisfied whatever the value of a_1. A further examination of these recurrence equations shows that they connect only alternate coefficients a_0, a_2, a_4, etc., and separately a_1, a_3, a_5, etc. One value of m can therefore give two independent solutions and therefore the complete solution. This will be examined further. If $m = 0$ $a_2 = \frac{1}{1\,.\,2}a_0$; $a_4 = \frac{1}{3\,.\,4}a_2 = \frac{1}{1\,.\,2\,.\,3\,.\,4}a_0$ and generally $a_r = \frac{1}{\lfloor r}a_0$ for even values of r.

Again $a_3 = \frac{1}{2\,.\,3}a_1$; $a_5 = \frac{1}{4\,.\,5}a_3 = \frac{1}{\lfloor 5}a_1$, and generally $a_r = \frac{1}{\lfloor r}a_1$, for odd values of r and so the solution is

$$y = a_0\left[1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^6}{\lfloor 6} \cdots\right] + a_1\left[x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} \cdots\right] \quad . \quad (470)$$

This solution is of course $y = a_0 \cosh x + a_1 \sinh x$ the known solution of equation 469.

Now, putting $m = 1$ does not give an independent solution. In this case in order that the coefficients of x^{m-1}, x^{m+1}, x^{m+3}, etc., shall vanish, the coefficients a_1, a_3, a_5, etc., must all be zero. The coefficients a_0, a_2, a_4, etc., can have real values. One has $a_2 = \frac{1}{2\,.\,3}a_0$; $a_4 = \frac{1}{4\,.\,5}a_2 = \frac{1}{\lfloor 5}a_0$ and generally $a_r = \frac{1}{\lfloor r+1}a_{r-2}$, giving as the solution

$$y = a_0\left[x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} \cdots\right] \quad . \quad . \quad (471)$$

This is seen to be a constant multiple of one of the original solutions. In general, if the indicial equation has two solutions m_1 and m_2 differing by an integer, and if a_1 becomes indeterminate when $m = m_1$, then this value of m will give the complete solution. The substitution of $m = m_2$ gives a solution which is a constant multiple of one of the separate solutions contained in the complete solution.

Legendre's Equation.

Legendre's equation plays an important part in Acoustical Engineering.* It is

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0 \quad . \quad . \quad (472)$$

where n is usually a positive integer.

It is more convenient from a practical point of view to express the solution of this equation as a series of *descending* powers of x. It is assumed that there is a solution of the form

$$y = a_0x^m + a_1x^{m-1} + a_2x^{m-2} \quad . \quad . \quad . \quad (473)$$

Then

$$n(n+1)y = n(n+1)x^m[a_0 + a_1x^{-1} + a_2x^{-2} + a_3x^{-3} + \ . \ . \ .]$$

$$-2x\frac{dy}{dx} = -2x^m[ma_0 + (m-1)a_1x^{-1} + (m-2)a_2x^{-2} + \ . \ . \ .]$$

$$\frac{d^2y}{dx^2} = x^m[m(m-1)a_0x^{-2} + (m-1)(m-2)a_1x^{-3} + \ . \ . \ .]$$

$$-x^2\frac{d^2y}{dx^2} = -x^m[m(m-1)a_0 + (m-1)(m-2)a_1x^{-1} + \ . \ . \ .]$$

Equating the coefficient of x^m to zero gives

$$a_0[n(n + 1) - 2m - m(m - 1)] = 0 \quad . \quad . \quad (474)$$

or

$$(m - n)(m + n + 1) = 0$$

so that $m = n$ or $-(n + 1)$. Taking the case $m = n$ gives on equating the coefficients of the various powers of x to zero

$$a_1 = a_3 = a_5 \ . \ . \ . = 0\,; \quad a_2 = -\frac{n(n - 1)}{2(2n - 1)}a_0$$

$$a_4 = \frac{n(n - 1)(n - 2)(n - 3)}{2 \ . \ 4 \ . \ (2n - 1)(2n - 3)}a_0, \text{ etc.}$$

* See McLachlan, *Loud Speakers.*

and so the first solution of equation 472 is

$$y_1 = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}x^{n-4} \ldots\right] \quad (475)$$

If a_0 is made equal to $\dfrac{1\,.\,3\,.\,5\,.\,.\,.\,(2n-1)}{\lfloor n} = \dfrac{\lfloor 2n}{2^n \lfloor n \lfloor n}$ the *First Legendre Function*, or *Surface Zonal Harmonic*, of the first kind of order n is obtained. It is denoted by $P_n(x)$ and is

$$P_n(x) = \frac{\lfloor 2n}{2^n \lfloor n \lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)}x^{n-4} \ldots\right] \quad (476)$$

This is a solution of equation 472. If n is half a positive integer some of the terms in equation 476 become infinite and this solution no longer exists. Usually in practical problems n is a positive integer. Then after a certain limited number of terms all subsequent terms become zero. In this case $P_n(x)$ is a polynomial—a *Legendre polynomial* of the first kind.

$P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2 - 1)$;
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$; $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$; etc.

The other solution to equation 472 is obtained by putting $m = -(n+1)$. Proceeding in an exactly similar way and giving to the arbitrary constant its conventional value the *Second Legendre Function*, or *Surface Zonal Harmonic*, of the second kind of order n is obtained, namely

$$Q_n(x) = \frac{2^n \lfloor n \lfloor n}{\lfloor 2n+1}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4\,.\,(2n+3)(2n+5)}x^{-n-5} + \ldots\right] \quad (477)$$

This series is convergent if $x < -1$ or > 1, divergent if x is between -1 and 1.

CHAPTER XVII

BESSEL FUNCTIONS

In the preceding chapter the method of obtaining a solution of a second order equation in the form of a series has been given. Because of its importance, Legendre's equation received some attention. A still more important equation from an engineering standpoint is Bessel's equation. In 1824 Bessel, while investigating planetary motion, obtained an integral which he defined as

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\,(n\theta - x\sin\theta)d\theta \quad . \quad . \quad (478)$$

$J_n(x)$ is known as a *Bessel function of the first kind, of order n.* It can be shown that it is a solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad . \quad . \quad . \quad (479)$$

This is known as Bessel's differential equation. It is often commonly written as (multiplying by x^2)

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad . \quad . \quad . \quad (480)$$

Another form is important from a practical point of view. In deducing a practical Bessel equation it is unlikely that the coefficient of y in equation 479 will be in the form $1 - \frac{n^2}{x^2}$ but will be of the form $a^2 - \frac{n^2}{x^2}$. The independent variable has to be changed to transform the equation into the standard form. Suppose in equation 479 one substitutes $x = kt$, then

$$\frac{dy}{dx} = \frac{1}{k}\frac{dy}{dt} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{k^2}\frac{d^2y}{dt^2}$$

so that equation 479 becomes

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(k^2 - \frac{n^2}{t^2}\right)y = 0 \quad . \quad . \quad . \quad (481)$$

The two equations 479 and 481 are of course identical and the solutions are identical namely $J_n(x)$ or $J_n(kt)$. The second form is the more useful being the more general. $J_n(kt)$ is a solution of equation 481.

Bessel's Equation of Zero Order.

The equation of zero order is of most common occurrence and will first be considered. It is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \quad . \quad . \quad . \quad . \quad (482)$$

Assuming a solution of the form of equation 460, substitution in equation 482 gives as the coefficients of the various powers of x

$$\begin{aligned} &x^{m-2}\,;\ a_0[m(m-1)+m] = a_0m^2 \\ &x^{m-1}\,;\ a_1[(m+1)m+m+1] = a_1(m+1)^2 \\ &x^{m}\,;\ a_2(m+2)^2 + a_0 \\ &x^{m+1}\,;\ a_3(m+3)^2 + a_1 \\ &x^{m+r}\,;\ a_{r+2}(m+r+2)^2 + a_r \end{aligned}$$

Since these are all zero then, if a_0 is to have a real value, m must be zero, and therefore a_1, a_3, a_5, etc., must all be zero. The first solution is therefore

$$y_1 = a_0x^m\left\{1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\right\} \quad . \quad . \quad (483)$$

subject to m being given the correct value.

If m is made zero and a_0 unity the solution is the *Bessel function of the first kind* of order zero

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(\underline{|}\,2)^2} - \frac{x^6}{2^6(\underline{|}\,3)^2} + \ldots$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{(x/2)^{2r}}{(\underline{|}\,r)^2} \quad . \quad . \quad (484)$$

This series is absolutely convergent for all values of x whether x is real or complex.

A second solution is required to this equation. Various solutions are possible ; one due to Neumann can be found by a

simple extension of the method already employed. If equation 483 be substituted in 482 it gives, when $a_0 = 1$,

$$\frac{d^2y_1}{dx^2} + \frac{1}{x}\frac{dy_1}{dx} + y_1 = m^2x^{m-2} \quad . \quad . \quad . \quad (485)$$

If this equation be differentiated with respect to m it becomes, if $q = \frac{\partial y_1}{\partial m}$,

$$\frac{d^2q}{dx^2} + \frac{1}{x}\frac{dq}{dx} + q = mx^{m-2}(2 + m\log x) \quad . \quad . \quad (486)$$

That is to say q is a solution of

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = mx^{m-2}(2 + m\log x) \quad . \quad . \quad (487)$$

But $q = \frac{\partial y_1}{\partial m}$ is, from equation 483,

$$\frac{\partial y_1}{\partial m} = y_1\log x + x^m\left\{\frac{x^2}{(m+2)^2}\cdot\frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \ldots\right\} \quad . \quad . \quad (488)$$

Now equation 482 whose second solution is required is identical with equation 487 when $m = 0$ and so the second solution is

$$y_2 = \left(\frac{\partial y_1}{\partial m}\right)_{m=0}$$

$$= y_1\log x + \left\{\frac{x^2}{2^2} - \frac{x^4}{2^4(\underline{|2})^2}\left(1+\frac{1}{2}\right) + \frac{x^6}{2^6(\underline{|3})^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)\right\} \ldots$$

or

$$y_2 = N_0(x)$$

$$= J_0(x)\log x - \sum_{r=1}^{\infty}(-1)^r\frac{(x/2)^{2r}}{(\underline{|r})^2}\left(1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{r}\right) \quad . \quad . \quad (489)$$

This is *Neumann's Bessel function of the second kind* of zero order.

Another second solution due to Weber is

$$y = Y_0(x) = \frac{2}{\pi}\{\log(x/2) + \gamma\}J_0(x)$$

$$- \frac{2}{\pi}\sum_{r=1}^{\infty}(-1)^r\frac{(x/2)^{2r}}{(\underline{|r})^2}\left(1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{r}\right) \quad . \quad . \quad (490)$$

where γ is Euler's constant and is

$$\gamma = \underset{m \to \infty}{\text{Limit}} \left[1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{m} - \log m \right] \doteqdot 0{\cdot}5772 \ldots \quad (491)$$

Since

$$Y_0(x) = \frac{2}{\pi}\{N_0(x) - \log (2 - \gamma) J_0(x)\} \quad \ldots \quad (492)$$

it is evident that $Y_0(x)$ is a solution if $N_0(x)$ is. $Y_0(x)$ is the generally accepted second solution. Hence the complete solution of the Bessel equation of order zero is

$$y = AJ_0(x) + BY_0(x) \quad \ldots \quad (493)$$

Bessel's Equation of Order n.

The equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad \ldots \quad (479)$$

Assuming as before that there is a solution of the form of equation 460 then the successive coefficients of powers of x are

$$\begin{aligned}
&x^{m-2}\,; \quad a_0[m(m-1) + m - n^2] = a_0(m^2 - n^2) \\
&x^{m-1}\,; \quad a_1[(m+1)^2 - n^2] \\
&x^{m}\,; \quad a_2[(m+2)^2 - n^2] + a_0 \\
&x^{m+r}\,; \quad a_{r+2}[(m+r+2)^2 - n^2] + a_r
\end{aligned}$$

Putting the coefficient of x^{m-2} equal to zero gives $m = \pm n$. Taking $m = n$ then, since $a_0 \neq 0$, $a_1 = a_3 = a_5 \ldots = 0$

$$a_{r+2} = -\frac{a_r}{(2n + r + 2)(r + 2)}$$

giving

$$a_2 = -\frac{1}{2(2n+2)}a_0\,; \quad a_4 = \frac{1}{2 \cdot 4(2n+2)(2n+4)}a_0, \text{ etc.}$$

and

$$y_1 = a_0 x^n \left[1 - \frac{(x/2)^2}{n+1} + \frac{(x/2)^4}{\underline{2}(n+1)(n+2)} - \frac{(x/2)^6}{\underline{3}(n+1)(n+2)(n+3)} + \ldots \right] \quad \ldots \quad (494)$$

A second solution is given by $m = -n$; again $a_1 = a_3 = a_5 = 0$

$$a_{r+2} = \frac{a_r}{(r+2)(2n-r-2)}$$

giving

$$a_2 = \frac{1}{2(2n-2)}a_0\,; \quad a_4 = \frac{1}{2\,.\,4(2n-2)(2n-4)}a_0, \text{ etc.}$$

and

$$y_2 = b_0 x^{-n}\left[1 + \frac{(x/2)^2}{n-1} + \frac{(x/2)^4}{\underline{2}(n-1)(n-2)} \cdots\right] \quad . \quad . \quad (495)$$

If in the value of y_1 one puts $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$ * then the value of y_1 is denoted by $J_n(x)$ and is

$$y_1 = J_n(x) = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \frac{(x/2)^2}{\Gamma(n+2)} + \frac{(x/2)^4}{\underline{2}\Gamma(n+3)} \cdots\right] \quad (496)$$

or

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{(x/2)^{n+2r}}{\underline{r}\Gamma(n+r+1)} \quad . \quad . \quad (497)$$

If in the value of y_2 one puts $b_0 = \dfrac{1}{2^{-n}\Gamma(-n+1)}$ then one obtains

$$y_2 = \left(\frac{x}{2}\right)^{-n}\left[\frac{1}{\Gamma(-n+1)} - \frac{(x/2)^2}{\Gamma(-n+2)} + \frac{(x/2)^4}{\underline{2}\Gamma(-n+3)} - \cdots\right] \quad (498)$$

Comparing equations 496 and 498, it is seen that the latter can be derived from the former by changing n to $-n$. It is denoted by $J_{-n}(x)$. When n is not an integer the series 496 and 498 are not linearly dependent and $J_{-n}(x)$ is another solution of equation 479. The complete solution is therefore

$$y = AJ_n(x) + BJ_{-n}(x) \quad . \quad . \quad . \quad . \quad (499)$$

When n is an integer further examination is necessary. $J_n(x)$ has been defined as

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{(x/2)^{n+2r}}{\underline{r}\Gamma(n+r+1)} \quad . \quad . \quad (497)$$

* *Note.*—When the equations are equally true for integral and non-integral values of n they may be generalised by the use of gamma functions.

If n is a positive integer this may be written

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{(x/2)^{n+2r}}{\underline{|r}\,\underline{|n+r}}$$

Each term in this expansion has a finite value if n is a positive integer or zero. If however n is a negative integer equal to, say, $-p$ then $\Gamma(n+r+1)$ becomes $\Gamma(r+1-p)$. This is finite so long as $r \not< p$ but is infinite if $r < p$ (p. 138). It therefore follows that each term in the expansion for which r is less than p is zero. If r is put equal to $p+s$ the series becomes

$$J_{-p}(x) = \sum_{p+s=0}^{\infty}(-1)^{p+s} \frac{(x/2)^{p+2s}}{\underline{|p+s}\,\Gamma(s+1)}$$

But each term for which s is negative is zero and therefore the only effective terms in the series occur from $s = 0$ to $s = \infty$ and therefore

$$J_{-p}(x) = \sum_{s=0}^{\infty}(-1)^{p+s} \frac{(x/2)^{p+2s}}{\underline{|p+s}\,\underline{|s}} \quad . \quad . \quad (500)$$

$$= (-1)^p J_p(x)$$

or generally

$$J_{-n}(x) = (-1)^n J_n(x) \quad . \quad . \quad . \quad . \quad (501)$$

Hence when n is integral the solutions $J_n(x)$ and $J_{-n}(x)$ are not separate and distinct but one is a simple multiple (± 1) of the other. Equation 499 is no longer the complete solution.

A second solution, not linearly dependent upon $J_n(x)$ when n is integral has therefore to be determined. Before doing this a few properties of $J_n(x)$ will be considered. If $J_0(x)$ and $J_1(x)$ be written out it will be found on differentiating $J_0(x)$, term by term, that

$$\frac{dJ_0}{dx} = -J_1 \quad . \quad . \quad . \quad . \quad . \quad (502)$$

(*Note.*—When there is no doubt regarding the argument it is convenient to omit it, shortening $J_0(x)$ to J_0.) This is a particular case of a more general relation. From equation 497 one may write

$$x^n J_n = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2n+2r}}{2^{n+2r}\,\underline{|r}\,\Gamma(n+r+1)} \quad . \quad (503)$$

and $$\frac{d}{dx}[x^n J_n] = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2n+2r-1}}{2^{n+2r-1}\,\underline{|r}\,\Gamma(n+r)}$$

$$= x^n \sum_{r=0}^{\infty}(-1)^r \frac{(x/2)^{n-1+2r}}{\underline{|r}\,\Gamma(n+r)}$$

or

$$\frac{d}{dx}[x^n J_n] = x^n J_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (504)$$

Similarly

$$\frac{d}{dx}[x^{-n} J_n] = -x^{-n} J_{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (505)$$

If $n = 0$

$$\frac{d}{dx} J_0 \text{ or } J_0' = -J_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (506)$$

From equation 504 $x^n \frac{dJ_n}{dx} + nx^{n-1} . J_n = x^n J_{n-1}$

or

$$xJ_n' + nJ_n = xJ_{n-1} \quad . \quad . \quad . \quad . \quad (507)$$

and similarly from equation 505

$$xJ_n' - nJ_n = -xJ_{n+1} \quad . \quad . \quad . \quad . \quad (508)$$

giving $2nJ_n = x(J_{n-1} + J_{n+1})$

or

$$J_{n-1} + J_{n+1} = \frac{2}{x} nJ_n \quad . \quad . \quad . \quad . \quad (509)$$

$$J_{n-1} = \frac{2}{x} nJ_n - J_{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (510)$$

$$= \frac{2}{x} nJ_n - \frac{2}{x}(n+2)J_{n+2} + J_{n+3} \quad . \quad . \quad (511)$$

and ultimately

$$J_{n-1} = \frac{2}{x}[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} \ldots] \quad . \quad . \quad . \quad (512)$$

since it is evident that $J_\infty = 0$.

The Second Solution to the Equation of Order n, when n is Integral.

Obtaining a second solution to the equation involves no new principle. The method adopted is similar to that employed in

obtaining the second solution to the equation of zero order. The work is rather more laborious but is straightforward. More than one solution is of course possible, depending upon the arbitrary coefficient adopted and whether any constant multiple of the first solution is included in the second. Weber's solution which will be used is denoted by $Y_n(x)$ and is, when n is integral,

$$Y_n(x)=\frac{2}{\pi}\{\gamma+\log(x/2)\}J_n(x)-\frac{1}{\pi}\sum_{r=0}^{n-1}\frac{\underline{|n-r-1}}{\underline{|r}}\left(\frac{2}{x}\right)^{n-2r}$$

$$-\frac{1}{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{(x/2)^{2r+n}}{\underline{|r}\;\underline{|n+r}}\left\{1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{r}+1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{n+r}\right\} \quad (513)$$

When n is not integral

$$Y_n=\frac{J_n\cos n\pi-J_{-n}}{\sin n\pi} \quad . \quad . \quad . \quad . \quad (514)$$

The complete solution to the equation of order n is then

$$y=AJ_n(x)+BY_n(x) \quad . \quad . \quad . \quad . \quad (515)$$

The Modified Bessel Functions $I_n(x)$ and $K_n(x)$.

If in the equation

$$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=0 \quad . \quad . \quad (479a)$$

one writes $z=jx$ then $\frac{dy}{dz}=j^{-1}\frac{dy}{dx}$ and $\frac{d^2y}{dz^2}=-\frac{d^2y}{dx^2}$. Substituting and dividing by -1 one obtains

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{n^2}{x^2}\right)y=0 \quad . \quad . \quad . \quad (516)$$

This equation should be carefully compared with equation 479. The solutions of equations 479a and 516 are of course identical. That is to say, the solution of equation 516 is

$$y=AJ_n(jx)+BY_n(jx) \quad . \quad . \quad . \quad . \quad (517)$$

It is desirable for the solution to be expressed in terms of a real argument instead of an imaginary one. Substituting in equation 497 one obtains

$$J_n(jx)=j^n\sum_{r=0}^{\infty}\frac{(x/2)^{n+2r}}{\underline{|r}\;\Gamma(n+r+1)}$$

The modified function of the first kind of order n is defined as

$$I_n(x) = j^{-n}J_n(jx) = \sum_{r=0}^{\infty} \frac{(x/2)^{n+2r}}{\lfloor r \, \Gamma(n+r+1)} \quad . \quad (518)$$

In particular

$$I_0(x) = 1 + \left(\frac{x}{2}\right)^2 + \frac{(x/2)^4}{(\lfloor 2)^2} + \frac{(x/2)^6}{(\lfloor 3)^2} + \ldots . \quad (519)$$

The modified function of the second kind is not so simply connected with the unmodified function. When n is integral it is defined as

$$K_n(x) = \frac{1}{2}\sum_{r=0}^{n-1}(-1)^r \frac{\lfloor n-r-1}{\lfloor r}\left(\frac{2}{x}\right)^{n-2r} +$$

$$(-1)^{n+1}\sum_{r=0}^{\infty}\frac{(x/2)^{n+2r}}{\lfloor r \quad \lfloor n+r}\left\{\log\frac{x}{2} - \frac{1}{2}[\psi(r+1)+\psi(n+r+1)]\right\} \quad . \quad (520)$$

where $$\psi(r+1) = \left(1 + \frac{1}{2} + \frac{1}{3} \ldots \frac{1}{r}\right) - \gamma$$

$$\psi(n+r+1) = \left(1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n+r}\right) - \gamma$$

When n is not integral

$$K_n(x) = \frac{\frac{1}{2}\pi\{I_{-n}(x) - I_n(x)\}}{\sin n\pi} \quad . \quad . \quad . \quad (521)$$

In particular

$$K_0(x) = -\left(\gamma + \log\frac{x}{2}\right)I_0(x) + \sum_{r=1}^{\infty}\frac{(x/2)^{2r}}{(\lfloor r)^2}\left\{1 + \frac{1}{2} + \frac{1}{3} \ldots \frac{1}{r}\right\} \quad (522)$$

Tables of many of these functions are given by McLachlan (*Bessel Functions for Engineers*).

The solution of equation 516 is then

$$y = AI_n(x) + BK_n(x) \quad . \quad . \quad . \quad . \quad (523)$$

Integral Forms of Bessel Functions.

In a number of practical problems the Bessel functions occur in their integral forms, particularly $J_n(x)$ and $I_n(x)$.

These are simply obtained. $\varepsilon^{\frac{x}{2}(t-t^{-1})} = \varepsilon^{\frac{x}{2}t} \times \varepsilon^{-\frac{x}{2t}}$, and so when $t \neq 0$

$$\varepsilon^{\frac{x}{2}(t-t^{-1})} = \left[1 + \left(\frac{xt}{2}\right) + \frac{(xt/2)^2}{\lfloor 2} + \frac{(xt/2)^3}{\lfloor 3} \ldots\right] \times \left[1 - \left(\frac{x}{2t}\right) + \frac{(x/2t)^2}{\lfloor 2} - \frac{(x/2t)^3}{\lfloor 3} + \ldots\right]$$

The coefficient of t^n in the product is evidently

$$\frac{(x/2)^n}{\lfloor n} - \frac{(x/2)^{n+2}}{\lfloor n+1} + \frac{(x/2)^{n+4}}{\lfloor 2 \lfloor n+2} - \frac{(x/2)^{n+6}}{\lfloor 3 \lfloor n+3} + \ldots = J_n(x)$$

when n is integral.

As all integral exponents of t, both positive and negative, occur

$$\varepsilon^{\frac{x}{2}(t-t^{-1})} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) \ldots$$

or

$$\varepsilon^{\frac{x}{2}(t-t^{-1})} = J_0 + J_1(t-t^{-1}) + J_2(t^2+t^{-2}) + J_3(t^3-t^{-3}) \ldots \quad (524)$$

Put $t = \cos\theta + j\sin\theta$

then $t^{-1} = \cos\theta - j\sin\theta$

$t^2 = \cos 2\theta + j\sin 2\theta$; $t^{-2} = \cos 2\theta - j\sin 2\theta$ etc.

Substituting in equation 524 gives

$$\varepsilon^{jx\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta \ldots] + 2j[J_1\sin\theta + J_3\sin 3\theta \ldots] \quad (525)$$

But $\varepsilon^{jx\sin\theta} = \cos(x\sin\theta) + j\sin(x\sin\theta)$

and so

$$\cos(x\sin\theta) = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta \ldots] \quad (526)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta \ldots] \quad (527)$$

Multiplying both sides of equation 526 by $\cos n\theta$ and integrating between 0 and π one obtains

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = 2\int_0^\pi J_n\cos^2 n\theta\, d\theta$$ when n is an even integer since $\int_0^\pi \cos m\theta\cos n\theta\, d\theta$ with m and n both integral is zero unless $m = n$. $\int_0^\pi \cos^2 n\theta\, d\theta = \frac{\pi}{2}$ and hence *if n is even*

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta$$

Similarly multiplying both sides of equation 527 by $\sin n\theta$ and integrating one obtains *when n is odd*

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta$$

If n is an even integer $\int_0^{\pi} \cos(x \sin\theta) \cos n\theta \, d\theta = \pi J_n$

and $\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta = 0.$

If n is an odd integer $\int_0^{\pi} \cos(x \sin\theta) \cos n\theta \, d\theta = 0$

and $\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta = \pi J_n.$

Hence generally if n is a positive integer

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} [\cos n\theta \cos(x \sin\theta) + \sin n\theta \sin(x \sin\theta)] d\theta$$

or $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x \sin\theta) d\theta.$$

This is Bessel's original definition given in a slightly different form in equation 478.

Writing $\frac{\pi}{2} - \theta$ for θ in equation 525 one obtains

$$e^{jx\cos x} = J_0 - 2[J_2 \cos 2\theta - J_4 \cos 4\theta + J_6 \cos 6\theta \ldots] + 2j[J_1 \cos\theta - J_3 \cos 3\theta + J_5 \cos 5\theta \ldots] \quad (528)$$

$$\cos(z\cos\theta) = J_0 - 2[J_2 \cos 2\theta - J_4 \cos 4\theta + J_6 \cos 6\theta \ldots]$$

$$\sin(z\cos\theta) = 2[J_1 \cos\theta - J_3 \cos 3\theta + J_5 \cos 5\theta \ldots]$$

If in these one puts $\theta = 0$ one obtains

$$\cos z = J_0(z) - 2[J_2(z) - J_4(z) + J_6(z) \ldots]$$

and $$\sin z = 2[J_1(z) - J_3(z) + J_5(z) \ldots]$$

Definite Integrals for $J_0(x)$ and $I_0(x)$

In some physical problems definite integrals are obtained which are Bessel functions, often of zero order. If equation 525 is integrated between the limits 0 and 2π all the trigonometrical functions on the right-hand side disappear and one obtains

$$\int_0^{2\pi} e^{jx\sin\theta} d\theta = \int_0^{2\pi} J_0(x) \, d\theta = 2\pi J_0$$

Similarly

$$\int_0^{2\pi} e^{jx\cos\theta} d\theta = 2\pi J_0$$

Further from equation 528 noticing that $\cos 2\theta$, $\cos 4\theta$, etc., have the same number of positive and negative intervals from 0 to π

$$\int_0^{2\pi} e^{jx\cos\theta}\,d\theta = 2\int_0^{\pi} e^{jx\cos\theta}\,d\theta$$

and so

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{jx\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{jx\cos\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi} e^{jx\cos\theta}d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(x\cos\theta)d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-jx\sin\theta}d\theta$$

And writing $I_0(x) = J_0(jx)$ one obtains

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-x\cos\theta}d\theta = \frac{1}{\pi}\int_0^{\pi} e^{-x\cos\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-x\sin\theta}d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(jx\cos\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cosh(x\cos\theta)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(x\sin\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{x\sin\theta}d\theta$$

CHAPTER XVIII

PROBLEMS INVOLVING BESSEL FUNCTIONS, ETC.

It is found in a thermionic valve that when the grid potential is sufficiently negative for the anode current to be quite small the relation between the anode current and the grid voltage is practically exponential and one may write

$$i_a = Ae^{be_g}$$

where e_g is the grid potential.

If the steady potential on the grid is $-V$ and an alternating potential $v \sin \omega t$ is superposed it is desired to find the increase in the average anode current.

$$\begin{aligned} i_a &= Ae^{b(-V+v \sin \omega t)} \\ &= Ae^{-bV}e^{bv \sin \omega t} \end{aligned}$$

The steady anode current (no alternating potential) is $i_a' = Ae^{-bV} = B$ say, so that $i_a = Be^{bv \sin \omega t}$.

Writing $\omega t = \theta$ the average value of i_a is

$$\frac{1}{2\pi}\int_0^{2\pi} Be^{bv \sin \theta} d\theta = BI_0(bv)$$

The increase in anode current is $B[I_0(bv) - 1]$.

High-frequency Current in a Cylindrical Conductor.

When alternating current flows along a long cylindrical conductor the distribution of current is practically symmetrical so long as the return is not too close. It is perfectly symmetrical if the return is a hollow cylindrical conductor, enclosing the conductor considered, and concentric with it. The conductor will itself be assumed to be hollow, of internal radius b and external radius a; the return will be assumed to be cylindrical and of internal radius c. The conditions in the central conductor are represented in Fig. 85. The conductor

being long the equipotential surfaces may be assumed to be planes perpendicular to the axis. At a radius r the flux density is B and the flux *within the conductor* linking 1 cm. length of the core of radius r is ϕ. The current density at radius r is g. The total current in the core of radius r is I_r. The potential difference per unit length of the conductor is ε. The external flux linking 1 cm. length of the conductor is ϕ'. The permeability is μ and the resistivity ρ. B, ϕ, g and I_r vary both with r and in time.

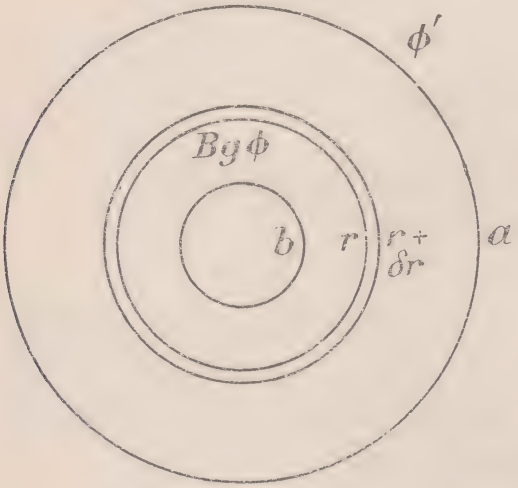


FIG. 85.

Then, working in absolute units,

$$\frac{\partial \phi}{\partial r} = -B = -\frac{2\mu}{r} I_r = -\frac{2\mu}{r}\int_b^r 2\pi g r\, dr \quad . \quad (529)$$

$$\varepsilon = g\rho + \frac{\partial \phi}{\partial t} + \frac{\partial \phi'}{\partial t}$$

$$\frac{\partial e}{\partial r} = 0 = \rho \frac{\partial g}{\partial r} + \frac{\partial^2 \phi}{\partial r\, \partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (530)$$

since the equipotentials are planes perpendicular to the axis and clearly $\frac{\partial \phi'}{\partial t}$ does not vary with r.

Equation 530 gives

$$\frac{\partial^2 \phi}{\partial r\, \partial t} = -\rho \frac{\partial g}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (531)$$

Differentiating equation 529 with respect to t gives

$$\frac{\partial^2 \phi}{\partial t\, \partial r} = -\frac{4\pi\mu}{r}\int_b^r r \frac{\partial g}{\partial t} dr \quad . \quad . \quad . \quad . \quad (532)$$

From equations 531 and 532

$$r \frac{\partial g}{\partial r} = \frac{4\pi\mu}{\rho}\int_b^r r \frac{\partial g}{\partial t} dr \quad . \quad . \quad . \quad . \quad . \quad (533)$$

Differentiating equation 533 with respect to r gives

$$r\frac{\partial^2 g}{\partial r^2} + \frac{\partial g}{\partial r} = \frac{4\pi\mu r}{\rho}\frac{\partial g}{\partial t}$$

or

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{4\pi\mu}{\rho}\frac{\partial g}{\partial t} = 0 \quad . \quad . \quad . \quad . \quad (534)$$

In the general case this equation may offer great difficulties in solution. We are particularly interested in the case when a steady sinoidal alternating current has been established. g may then be considered to be the vector current density; then $\frac{\partial g}{\partial t} = j\omega g$ (see p. 99) and the equation becomes

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{4\pi\mu}{\rho} j\omega g = 0$$

The partial differentials may now be written as complete differentials and the equation as

$$\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - jk^2 g = 0 \quad . \quad . \quad . \quad . \quad (535)$$

where $k^2 = 4\pi\mu\omega/\rho$.

The solution of equation 535 introduces some new functions of importance. Referring back to equation 516, if $n = 0$ this becomes

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - y = 0 \quad . \quad . \quad . \quad . \quad (536)$$

Its solution is

$$y = AI_0(x) + BK_0(x) \quad . \quad . \quad . \quad . \quad (537)$$

Similarly the solution of

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - k^2 y = 0 \quad . \quad . \quad . \quad . \quad (538)$$

is

$$y = AI_0(kx) + BK_0(kx) \quad . \quad . \quad . \quad . \quad (539)$$

It therefore follows that the solution of equation 535 is

$$g = AI_0(kr\sqrt{j}) + BK_0(kr\sqrt{j}) \quad . \quad . \quad . \quad (540)$$

Convenient series for I_0 and K_0 are, when x is small,

$$I_0(x) = 1 + \left(\frac{x}{2}\right)^2 + \frac{(x/2)^4}{(\lfloor 2)^2} + \frac{(x/2)^6}{(\lfloor 3)^2} \quad . \quad . \quad . \quad . \quad (519)$$

and $K_0(x) = I_0(x)[\log 2 - \gamma - \log x]$

$$+\left(\frac{x}{2}\right)^2 + (1 + \tfrac{1}{2})\frac{(x/2)^4}{(\underline{|2})^2} + (1 + \tfrac{1}{2} + \tfrac{1}{3})\frac{(x/2)^6}{(\underline{|3})^2} + \ldots \quad (541)$$

When x is big the equivalent series are

$$I_0(x) = \frac{\varepsilon^x}{\sqrt{2\pi x}}\left[1 + \frac{1^2}{8x} + \frac{1^2 \cdot 3^2}{\underline{|2} \cdot (8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{\underline{|3}(8x)^3} + \ldots\right] . \quad (542)$$

and

$$K_0(x) = \varepsilon^{-x}\sqrt{\frac{\pi}{2x}}\left[1 - \frac{1^2}{8x} + \frac{1^2 \cdot 3^2}{\underline{|2}(8x)^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{\underline{|3}(8x)^3} + \ldots\right] \quad (543)$$

Ber, bei, ker and kei Functions.

Four new functions are defined by the equations

$$I_0(x\sqrt{j}) = \text{ber}\, x + j\,\text{bei}\, x \quad . \quad . \quad . \quad (544)$$

$$K_0(x\sqrt{j}) = \text{ker}\, x + j\,\text{kei}\, x \quad . \quad . \quad . \quad (545)$$

ber = *Be*ssel *r*eal ; bei = *Be*ssel *i*maginary ; ker and kei are analogues. Simple substitution in equation 519 gives

$$\text{ber}\, x = 1 - \frac{(x/2)^4}{(\underline{|2})^2} + \frac{(x/2)^8}{(\underline{|4})^2} - \ldots \quad . \quad . \quad (546)$$

$$\text{bei}\, x = \left(\frac{x}{2}\right)^2 - \frac{(x/2)^6}{(\underline{|3})^2} + \frac{(x/2)^{10}}{(\underline{|5})^2} \quad . \quad . \quad . \quad . \quad (547)$$

Similar, but rather more complicated, expressions can be obtained for ker x and kei x. Asymptotic expressions when x is big are given later.

Solid Wire.

Since log 0 is infinite it follows that $K_0(x)$ is infinite when $x = 0$ and since the current density at the centre of the wire is not infinite it follows that, for the solid wire, B in equation 540 is zero. The appropriate solution is therefore

$$g = AI_0(kr\sqrt{j}) = A(\text{ber}\, kr + j\,\text{bei}\, kr) \quad . \quad . \quad (548)$$

Simple integration of the expansions enable the following important relations to be obtained

$$\int r\,\text{ber}\, kr\, dr = \frac{r}{k}\,\text{bei}'\, kr \quad . \quad . \quad . \quad . \quad (549)$$

$$\int r\,\text{bei}\, kr\, dr = -\frac{r}{k}\,\text{ber}'\, kr \quad . \quad . \quad . \quad (550)$$

$$\int r \ker kr\, dr = \frac{r}{k} \text{kei}' kr \quad . \quad . \quad . \quad . \quad (551)$$

$$\int r \text{ kei } kr\, dr = -\frac{r}{k} \ker' kr \quad . \quad . \quad . \quad (552)$$

The total current flowing along the wire is

$$I_a = 2\pi \int_0^a g r\, dr = 2\pi A \int_0^a r(\text{ber } kr + j \text{ bei } kr)\, dr$$

or

$$I_a = \frac{2\pi A a}{k}(\text{bei}' ka - j \text{ ber}' ka) \quad . \quad . \quad . \quad (553)$$

If ε' stands for that part of the potential difference which does not include the e.m.f. generated by external flux, that is

$$\varepsilon' = g\rho + \frac{\partial \phi}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (554)$$

At the surface $\phi = 0$ and

$$g = A(\text{ber } ka + j \text{ bei } ka) \quad . \quad . \quad . \quad . \quad (555)$$

Hence

$$\varepsilon' = A\rho(\text{ber } ka + j \text{ bei } ka) \quad . \quad . \quad . \quad . \quad (556)$$

The power per unit length is therefore (from equations 553 and 556)

$$= \frac{2\pi A^2 \rho a}{k}(\text{ber } ka \text{ bei}' ka - \text{bei } ka \text{ ber}' ka) \quad . \quad (557)$$

The magnitude of I_a^2 is

$$\frac{4\pi^2 A^2 a^2}{k^2}(\text{bei}'^2 ka + \text{ber}'^2 ka) \quad . \quad . \quad . \quad (558)$$

The resistance per unit length is therefore

$$R_f = \frac{\rho k}{2\pi a}\left(\frac{\text{ber } ka \text{ bei}' ka - \text{bei } ka \text{ ber}' ka}{\text{ber}'^2 ka + \text{bei}'^2 ka}\right) \quad . \quad (559)$$

The resistance at low frequency is $R_0 = \dfrac{\rho}{\pi a^2}$

and

$$\frac{R_f}{R_0} = \frac{ka}{2}\frac{W(ka)}{V(ka)} \quad . \quad . \quad . \quad . \quad . \quad (560)$$

where $W(ka) = \text{ber } ka \text{ bei}' ka - \text{bei } ka \text{ ber}' ka$

and $V(ka) = \text{ber}'^2 ka + \text{bei}'^2 ka$

When x is small

$$\frac{W(x)}{V(x)} = \frac{2}{x}\left[1 + \frac{1}{12}\left(\frac{x}{2}\right)^4 - \frac{1}{180}\left(\frac{x}{2}\right)^8 \ldots\right]^* \quad (561)$$

When x is big

$$\frac{W(x)}{V(x)} = \frac{1}{\sqrt{2}} + \frac{1}{2x} + \frac{3}{8\sqrt{2}x^2} \ldots \quad (562)$$

If one writes $\frac{ka}{2\sqrt{2}} = z$ then at high frequencies ($k \propto \sqrt{f}$)

$$\frac{R_f}{R_0} = \frac{ka}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{2ka} + \frac{3}{8\sqrt{2}k^2a^2} \ldots\right)$$

$$= \frac{ka}{2\sqrt{2}} + \frac{1}{4} + \frac{3 \times 2\sqrt{2}}{64ka} + \ldots$$

or

$$\frac{R_f}{R_0} = z + \frac{1}{4} + \frac{3}{64z} + \ldots \quad (563)$$

Rayleigh gave the approximation at high frequencies $\frac{R_f}{R_0} = z$; a second approximation by Howe was $\frac{R_f}{R_0} = z + \frac{1}{4}$.

Many of the functions involved in the calculation of high frequency resistance of wires have been computed by the Bureau of Standards, Washington. Some of the most important of these are reprinted in Russell (*Alternating Currents*).

For copper one may assume $\rho = 1{\cdot}7 \times 10^{-6}$ ohms/cm.^{3} or 1700 C.G.S. units so that $k = \sqrt{8\pi^2 f/1700} = 0{\cdot}215\sqrt{f}$.

Example.

In terms of the direct current resistance find the alternating current resistance of a copper wire 0·2 cm. diameter at frequencies of 10^2, 10^4 and 10^6.

(*a*) $f = 10^2$. Here equation 561 is applicable

$$a = 0{\cdot}1,\ k = 2{\cdot}15,\ ka = 0{\cdot}215$$

$$\frac{R_f}{R_0} = 1 + \frac{1}{12}\left(\frac{ka}{2}\right)^4 - \frac{1}{180}\left(\frac{ka}{2}\right)^8 \text{ etc.}$$

The increase in resistance is less than 1 part in 10,000.

* Russell, *Alternating Currents*, Vol. I. This and many other formulæ in this Chapter are taken from this book.

(*b*) $f = 10^4$, $ka = 2{\cdot}15$.

This case presents difficulty. Both formulæ 561 and 562 are applicable if sufficient terms are included. From the table given in Russell

$$\frac{x}{2} \cdot \frac{W(x)}{V(x)} = 1{\cdot}102 \text{ when } x = 2{\cdot}15$$

so that the resistance is increased by 10·2 per cent.

(*c*) $f = 10^6$, $ka = 21{\cdot}5$.

Here equation 562 is applicable giving

$$\frac{W(21{\cdot}5)}{V(21{\cdot}5)} = 0{\cdot}707 + 0{\cdot}023 + 0{\cdot}001 = 0{\cdot}731$$

$$\frac{R_f}{R_0} = \frac{21{\cdot}5}{2} \times 0{\cdot}731 = 7{\cdot}87.$$

Conditions at Very High Frequencies.

The expression high frequency must be understood in reference to the size, resistivity and permeability of the material. The relevant quantity is actually ka.

When ka is very large

$$R_f = \frac{\rho k}{2\pi a} \frac{W(ka)}{V(ka)} = \frac{\rho k}{2\pi a} \cdot \frac{1}{\sqrt{2}} \quad . \quad . \quad . \quad (564)$$

It is seen that under these circumstances the conductivity is proportional to the radius, as opposed to the square of the radius at low frequencies. Putting in the value of k gives

$$R_{ff} = \frac{1}{2\pi a} \sqrt{\frac{\rho^2 \cdot 8\pi^2 \mu f}{2\rho}}$$

or

$$R_{ff} = \frac{1}{a} \sqrt{\mu \rho f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (565)$$

At very high frequencies the following asymptotic expressions are nearly accurate:

$$\text{ber } x = X \cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \quad . \quad . \quad . \quad (566)$$

$$\text{bei } x = X \sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \quad . \quad . \quad . \quad (567)$$

$$\text{ber}' x = X \cos\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \quad . \quad . \quad . \quad (568)$$

$$\text{bei}' x = X \sin\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \quad . \quad . \quad . \quad (569)$$

where $X = \dfrac{1}{\sqrt{2\pi x}} e^{x/\sqrt{2}}$

The impedance of a wire per unit length is from equations 553 and 556

$$\frac{\varepsilon'}{I_a} = \frac{k\rho}{2\pi a}\left[\frac{\text{ber } ka + j \text{ bei } ka}{\text{bei}' ka - j \text{ ber}' ka}\right]$$

At very high frequencies this becomes, from equations 566–9,

$$\frac{\varepsilon'}{I_a} = \frac{k\rho}{2\pi a}\left[\frac{\cos\theta + j \sin\theta}{\sin\left(\theta + \frac{\pi}{4}\right) - j \cos\left(\theta + \frac{\pi}{4}\right)}\right]$$

where $\theta = \dfrac{ka}{\sqrt{2}} - \dfrac{\pi}{8}$

or

$$\frac{\varepsilon'}{I_a} = \frac{k\rho}{2\pi a} \cdot \frac{je^{j\theta}}{e^{j(\theta+\frac{\pi}{4})}} = \frac{k\rho}{2\pi a} \cdot \frac{1}{\sqrt{2}}(1 + j) \quad . \quad . \quad (570)$$

Hence $R_f = \dfrac{k\rho}{2\sqrt{2}\pi a}$ and $X_f = \dfrac{k\rho}{2\sqrt{2}\pi a}$

The expression for the resistance is that already obtained (equation 564). It is seen that the reactance is equal to the resistance. The voltage therefore leads the current by 45°.

The Hollow Conductor.

When the conductor is hollow it is necessary to write the solution

$$g = AI_0(kr\sqrt{j}) + BK_0(kr\sqrt{j}) \quad . \quad . \quad . \quad (540)$$

or

$$g = A(\text{ber } kr + j \text{ bei } kr) + B(\text{ker } kr + j \text{ kei } kr) \quad . \quad (571)$$

then

$$I_r = \frac{2\pi Ar}{k}(\text{bei}' kr - j \text{ ber}' kr) + \frac{2\pi Br}{k}(\text{ker}' kr - j \text{ kei}' kr) \quad (572)$$

$I_b = 0$ giving B in terms of A

$$e' = A\rho(\text{ber } ka + j \text{ bei } ka) + B\rho(\text{ker } ka + j \text{ kei } ka) \quad . \quad . \quad (573)$$

$$\frac{B}{A} = -\frac{\text{bei}' kb - j \text{ ber}' kb}{\text{kei}' kb - j \text{ ker}' kb}$$

At very high values of x

$$\ker x = Y \cos\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \quad . \quad . \quad . \quad . \quad (574)$$

$$\text{kei}\, x = -Y \sin\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \quad . \quad . \quad . \quad (575)$$

$$\ker' x = -Y \cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \quad . \quad . \quad . \quad (576)$$

$$\text{kei}' x = Y \sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \quad . \quad . \quad . \quad . \quad (577)$$

where $Y = \sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}}$.

It is not proposed to detail the whole of the working which is straightforward but laborious. The final result obtained is

$$R_{ff} = \frac{k\rho}{2\sqrt{2}\pi a} \cdot \frac{\sinh \sqrt{2}k(a-b) + \sin \sqrt{2}k(a-b)}{\cosh \sqrt{2}k(a-b) - \cos \sqrt{2}k(a-b)} \quad . \quad (578)$$

Distribution of Current in the Solid Conductor.

The current density throughout the conductor is given by equation 548

$$g = A(\text{ber}\, kr + j\, \text{bei}\, kr)$$

The magnitude of the current is proportional to

$$M_0 = \sqrt{\text{ber}^2 kr + \text{bei}^2 kr} \quad . \quad . \quad . \quad (579)$$

M_0 increases with kr, at first slowly, but faster later. For instance $M_0(0) = 1$, $M_0(1) = 1{\cdot}016$, $M_0(2) = 1{\cdot}229$. The phase of g changes with kr.

It follows that the current density is greatest at the surface of the conductor, decreasing towards the centre. At high frequencies where kr may have high values this effect is very marked. Thus if kr is 10, the current density at 0·8 of the radius is only 27 per cent. of its value at the surface and lags behind by 81°. At 0·6 of the radius the current density is only 8 per cent. of its value at the surface and lags behind the surface current by 162°. Slightly nearer the centre the current is in exactly opposite phase to the current at the surface. In fact a tube of the right thickness can actually have less resistance than a solid conductor of the same overall diameter.

Eddy Current Loss in the Core of a Solenoid.

When a mass of metal is subjected to a varying magnetic force, eddy currents are generated in it tending to prevent change of the magnetic field. If the magnetic force is alternating the variation in the magnetic field may, due to eddy currents in the surface layers, be nearly inhibited in the interior of the metal. This effect is very pronounced in ferromagnetic materials. For this reason, chiefly, the core of a dynamo must be laminated, otherwise the penetration of flux into the core under working conditions would not be sufficient to generate a reasonable voltage in the conductors in the slots. Due to eddy currents there is of course energy wasted in the metal. This energy may be very considerable. In the case of iron or steel the energy lost is small if the metal is finely laminated or if the section is large and the metal is unlaminated. For a given magnetising force the energy lost is a maximum for a certain degree of coarse lamination. In the induction furnace it is desirable that the energy absorbed per cubic centimetre should be as high as possible. The optimum condition calls for a certain degree of division of the material depending upon its permeability and its conductivity and upon the frequency.

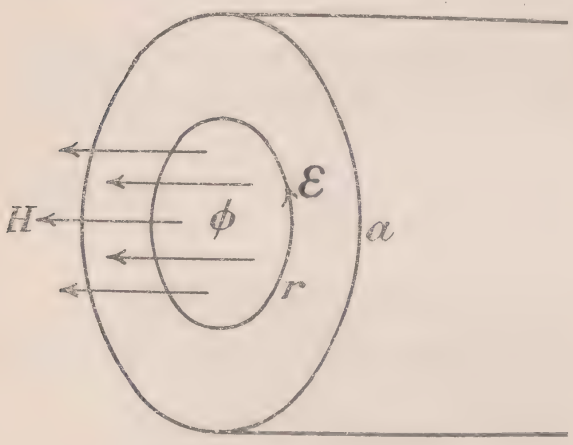


Fig. 86.

The case considered will be that of a long rod within a solenoid, the magnetic force being directed along the axis and being uniform throughout the length.

At a radius r (Fig. 86) the magnetic force is H. The total flux within this circle of radius r is

$$\phi = \int_0^r \mu H \,.\, 2\pi r \, dr \quad . \quad . \quad . \quad . \quad . \quad (580)$$

Due to variation of ϕ there is an e.m.f. ε generated in the circle of radius r defined by

$$\varepsilon = -\frac{\partial \phi}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (581)$$

and in consequence there is an eddy current similarly directed of intensity

$$g = \frac{\varepsilon}{2\pi r\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (582)$$

Passing outwards from r to $r + \delta r$ the encircling current per unit length *diminishes* by $g\delta r$ and the change in the magnetic force δH is equal to 4π times this or

$$\frac{\partial H}{\partial r} = -4\pi g \quad . \quad . \quad . \quad . \quad . \quad (583)$$

From equations 580 and 581 one obtains

$$\varepsilon = -2\pi\mu\int_0^r r\frac{\partial H}{\partial t}dr \quad . \quad . \quad . \quad . \quad (584)$$

and then using equation 582 gives

$$g = -\frac{\mu}{\rho r}\int_0^r r\frac{\partial H}{\partial t}dr \quad . \quad . \quad . \quad . \quad . \quad (585)$$

But from equation 583, $g = -\frac{1}{4\pi}\frac{\partial H}{\partial r}$ and so

$$r\frac{\partial H}{\partial r} = \frac{4\pi\mu}{\rho}\int_0^r r\frac{\partial H}{\partial t}dr \quad . \quad . \quad . \quad . \quad (586)$$

Differentiating with respect to r one obtains

$$r\frac{\partial^2 H}{\partial r^2} + \frac{\partial H}{\partial r} = \frac{4\pi\mu r}{\rho}\frac{\partial H}{\partial t} \quad . \quad . \quad . \quad . \quad (587)$$

Again, as in the case of equation 534, this equation cannot be solved unless the nature of $\frac{\partial H}{\partial t}$ is known. When the flux is alternating and sinoidal equation 587 may be regarded as a vector equation, $j\omega H$ being written for $\frac{\partial H}{\partial t}$. Complete differentials may then be used and the equation becomes

$$\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr} - jk^2H = 0 \quad . \quad . \quad . \quad . \quad (588)$$

where $k^2 = 4\pi\mu\omega/\rho$ as before. For the solid rod the solution is

$$H = AI_0(kr\sqrt{j}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (589)$$

or

$$H = A(\text{ber } kr + j \text{ bei } kr) \quad . \quad . \quad . \quad (590)$$

At the surface of the rod

$$H_0 = A(\text{ber } ka + j \text{ bei } ka) \quad . \quad . \quad . \quad (591)$$

and hence

$$H = H_0\left[\frac{\text{ber } kr + j \text{ bei } kr}{\text{ber } ka + j \text{ bei } ka}\right] \quad . \quad . \quad . \quad (592)$$

From equation 580 one obtains

$$\phi = 2\pi\mu A\int_0^r r(\text{ber } kr + j \text{ bei } kr)dr$$

or

$$\phi = \frac{2\pi\mu Ar}{k}(\text{bei}' \, kr - j \text{ ber}' \, kr)$$

or

$$\phi_0 = 2\pi\mu H_0\frac{a}{k}\left[\frac{\text{bei}' \, ka - j \text{ ber}' \, ka}{\text{ber } ka + j \text{ bei } ka}\right] \quad . \quad . \quad (593)$$

and

$$|\phi_0| = \pi a^2\mu H_0\left[\frac{2}{ka} \cdot \frac{\sqrt{\text{bei}'^2 \, ka + \text{ber}'^2 \, ka}}{\sqrt{\text{ber}^2 \, ka + \text{bei}^2 \, ka}}\right] \quad . \quad (594)$$

Russell writes

$$X(x) = \text{ber}^2 \, x + \text{bei}^2 \, x \quad . \quad . \quad . \quad . \quad (595)$$

and

$$V(x) = \text{bei}'^2 \, x + \text{ber}'^2 \, x \quad . \quad . \quad . \quad . \quad (596)$$

McLachlan writes $M_0(x) = \sqrt{\text{ber}^2 \, x + \text{bei}^2 \, x}$

and

$$M_1(x) = \sqrt{\text{bei}'^2 \, x + \text{ber}'^2 \, x}$$

For large values of x

$$\frac{V}{X} = 1 - \frac{1}{x\sqrt{2}} + \frac{1}{4x^2} + \frac{3}{8\sqrt{2}x^3} \quad . \quad . \quad . \quad . \quad (597)$$

Before proceeding further, an example will be worked out. The case of steel can of course only be computed approximately since the permeability varies. The general trend can, however, be followed if a reasonable average value be assumed. Reasonable values are $\mu = 10^3$, $\rho = 10^{-5}$ ohms per cm. cube or 10^4 C.G.S. units. Then

$$k = 2{\cdot}81\sqrt{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad (598)$$

Conditions will be examined for a core 10 cm. diameter at a frequency of 50. Then $k = 19{\cdot}86$ and $ka = 99{\cdot}3$. For such high values of kr as are likely to be involved near the surface

one may substitute from equations 566 to 569 when equation 592 becomes

$$H = H_0 \sqrt{\frac{2\pi ka}{2\pi kr}} \cdot \frac{e^{kr/\sqrt{2}}}{e^{ka/\sqrt{2}}} \cdot \frac{\cos\left(\frac{kr}{\sqrt{2}} - \frac{\pi}{8}\right) + j \sin\left(\frac{kr}{\sqrt{2}} - \frac{\pi}{8}\right)}{\cos\left(\frac{ka}{\sqrt{2}} - \frac{\pi}{8}\right) + j \sin\left(\frac{ka}{\sqrt{2}} - \frac{\pi}{8}\right)} \qquad (599)$$

The last member of this product indicates that the phase of the magnetising force falls back by an angle $\frac{k}{\sqrt{2}}(a - r)$ in passing from the surface to a distance $(a - r)$ below the surface. The phase is completely reversed when

$$\frac{k}{\sqrt{2}}(a - r) = \pi \text{ giving } a - r = 0{\cdot}223 \text{ cm.}$$

One millimetre below the surface

$$|H| = |H_0| \sqrt{\frac{5}{4{\cdot}9}} e^{19{\cdot}86 \times 0{\cdot}1/\sqrt{2}} = 0{\cdot}248 |H_0|$$

The effective permeability is from equation 594

$$\mu_e = \mu \left[\frac{2}{ka} \sqrt{\frac{V(ka)}{X(ka)}} \right] = \mu \cdot \frac{2}{ka} \cdot \frac{M_1(ka)}{M_0(ka)} \qquad (600)$$

For the case being considered

$$\mu_e = \mu \times \frac{2}{99{\cdot}3} \times 0{\cdot}993 = 20$$

The total flux is only 2 per cent. of what it would be with direct current of the same value. It is seen that at high frequencies equation 600 degenerates to

$$\mu_e = \frac{2}{ka}\mu \qquad (601)$$

At 50,000 cycles per second, $k = 629$ and $ka = 3145$ whence μ_e becomes 0·635 or less than that of air.

Energy Expended.

From equations 583 and 590

$$g = -\frac{Ak}{4\pi}(\text{ber}' kr + j \,\text{bei}' kr) \qquad (602)$$

The loss per centimetre length of the rod in the ring between r and $r + \delta r$ is

$$|g^2| \cdot \rho 2\pi r\, \delta r = \frac{|A^2|\rho r k^2}{8\pi}(\text{ber}'^2\, kr + \text{bei}'^2\, kr)\delta r$$

and the total loss per centimetre length is

$$W_1 = \frac{|A^2|\rho k^2}{8\pi}\int_0^a r(\text{ber}'^2\, kr + \text{bei}'^2\, kr)dr$$

It can be shown that

$$\int_0^a r(\text{ber}'^2\, kr + \text{bei}'^2\, kr)\, dr = \frac{a}{k}(\text{ber}\, ka\, \text{ber}'\, ka + \text{bei}\, ka\, \text{bei}'\, ka)$$

and so $W_1 = \dfrac{|A^2|\rho ka}{8\pi}(\text{ber}\, ka\, \text{ber}'\, ka + \text{bei}\, ka\, \text{bei}'\, ka)$

or $$W_1 = \frac{H_0^2\rho ka}{8\pi}\left(\frac{\text{ber}\, ka\, \text{ber}'\, ka + \text{bei}\, ka\, \text{bei}'\, ka}{\text{ber}^2\, ka + \text{bei}^2\, ka}\right)$$

The loss per cubic centimetre in ergs per second is

$$W_{c.c.} = \frac{H_0^2\rho k}{8\pi^2 a} \cdot \frac{Z(ka)}{X(ka)} \quad . \quad . \quad . \quad . \quad (603)$$

where $Z(x) = \text{ber}\, x\, \text{ber}'\, x + \text{bei}\, x\, \text{bei}'\, x$.

Alternative expressions are

$$W_{c.c.} = \frac{H_0^2\rho k^2}{16\pi^2}\left[\frac{2}{ka} \cdot \frac{Z(ka)}{X(ka)}\right]$$

and $$W_{c.c.} = \frac{H_0^2\mu f}{ka} \cdot \frac{Z(ka)}{X(ka)}$$

If the frequency is fixed then the best diameter of rods to absorb as much energy as possible per cubic centimetre is given by

$$\frac{1}{ka} \cdot \frac{Z(ka)}{X(ka)} \text{ to be a maximum.}$$

For a fixed frequency the expression above has a maximum when ka is about 2·5. Its value is then about 0·190. Then for steel rods at a frequency of 50, $k = 19{\cdot}86$ and so $a = 0{\cdot}126$ cm. That is to say at 50 cycles per second rods of approximately $\frac{1}{4}$ cm. diameter will heat more rapidly than thinner or thicker rods.

Conditions are different if the frequency may be varied. For big values of ka, $\frac{1}{ka} \cdot \frac{Z(ka)}{X(ka)} \doteqdot \frac{1}{\sqrt{2}ka}$

or $$\frac{1}{ka} \cdot \frac{Z(ka)}{X(ka)} \doteqdot \frac{c}{\sqrt{f}}$$

Hence $$\frac{f}{ka} \cdot \frac{Z(ka)}{X(ka)} \doteqdot c\sqrt{f} \quad . \quad . \quad . \quad . \quad (604)$$

or the heating is proportional to the square root of the frequency.

Very High Frequencies.

At very high frequencies equations 535 and 588 may be considerably simplified by considering r big. Big is of course a relative term and here it means great compared with the effective depth of penetration of current in the one case and of flux in the other (this distinction is really spurious, for where there is flux there is current and vice versa). These equations then become

$$\frac{d^2g}{dr^2} - jk^2g = 0 \quad . \quad . \quad . \quad . \quad . \quad (605)$$

and

$$\frac{d^2H}{dr^2} - jk^2H = 0 \quad . \quad . \quad . \quad . \quad . \quad (606)$$

The solutions of each are of course similar. The former only will be briefly considered. Writing $x = a - r$ where x is the depth below the surface equation 605 becomes

$$\frac{d^2g}{dx^2} - jk^2g = 0 \quad . \quad . \quad . \quad . \quad . \quad (607)$$

or $$g = Ae^{-k\sqrt{j}x} + Be^{k\sqrt{j}x}$$

When x is big $e^{k\sqrt{j}x}$ tends to infinity, hence $B = 0$ and the applicable solution is

$$g = Ae^{-k\sqrt{j}x} = Ae^{-\frac{k}{\sqrt{2}}(1+j)x}$$

or $$g = g_0{}^{-\alpha x}(\cos \alpha x - j \sin \alpha x) \quad . \quad . \quad . \quad (608)$$

where $\alpha^2 = \frac{k^2}{2} = \frac{2\pi\mu\omega}{\rho}$ and g_0 = current density at the surface.

At a distance $\alpha x = 1$ the current density falls to $\frac{1}{e}$ of its value

at the surface and its phase falls back by 1 radian. The total current per centimetre of circumference of the conductor is

$$I_1 = g_0 \int_0^\infty e^{-\alpha x}(\cos \alpha x - j \sin \alpha x)\, dx$$

$$= g_0 \int_0^\infty e^{-\alpha(1+j)x} dx = \left[\frac{g_0 e^{-\alpha(1+j)x}}{-\alpha(1+j)} \right]_0^\infty$$

or

$$I_1 = \frac{g_0}{\alpha(1+j)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (609)$$

$$|I_1| = \frac{g_0}{\sqrt{2}\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (610)$$

This current clearly lags behind the surface current by 45°. Were the surface density and surface phase to be maintained the current I_1 could be accounted for by current in a skin l_1 thick ; $l_1 = \frac{1}{\sqrt{2}\alpha} = \frac{1}{k}$. This is sometimes termed the depth of penetration. The total axial current along the conductor, irrespective of whether it is travelling forwards or backwards is

$$I_2 = g_0 \int_0^\infty e^{-\alpha x} dx = \frac{g_0}{\alpha} \quad . \quad . \quad . \quad . \quad (611)$$

This is $\sqrt{2}$ times the net current I_1. This current could be accounted for by the surface density being maintained to a depth $l_2 = \frac{1}{\alpha}$; $l_2 = \sqrt{2} l_1$. By some writers l_2 instead of l_1 is termed the depth of penetration. These expressions, though inexact, are useful from a practical point of view, for they give a single quantitative idea of the rate of attenuation as one proceeds inwards from the surface.

Resistance of a Wide Parallel Strip.

The resistance of a wide parallel strip and the penetration of flux into a lamination at high frequencies present exactly similar problems. Both will be considered. The wide parallel strip is represented in Fig. 87 by its cross-section. Its width is b, its thickness $2a$; $b >> 2a$. The current flows in a direction represented by the normal to the paper. At a distance x from the mid plane OO' the current density is g, the intensity of the magnetic field is H (directed along BC, or CB),

the flux per unit length within the conductor, and to the right of BC, is ϕ ; the external flux to the right of the conductor is ϕ' ; the current within the section $OBCO'$ is I_x ; the potential difference per unit length of the conductor is ε. Proceeding round the path $ABCD$ the current enclosed is $2I_x$ and therefore

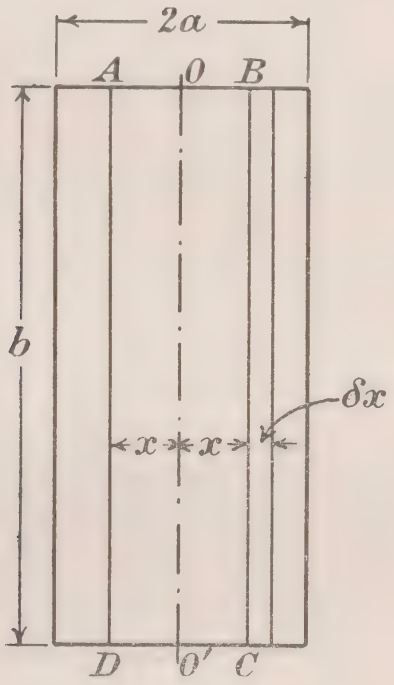


Fig. 87.

the m.m.f. is $8\pi I_x$. This is equal to $H \times 2b$. Hence

$$\frac{\partial \phi}{\partial x} = -B = -\mu H = -\frac{4\pi\mu}{b} I_x = -4\pi\mu \int_0^x g\, dx \quad (612)$$

$$\varepsilon = g\rho + \frac{\partial \phi}{\partial t} + \frac{\partial \phi'}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (613)$$

$$\frac{\partial \varepsilon}{\partial x} = 0 = \rho \frac{\partial g}{\partial x} + \frac{\partial^2 \phi}{\partial x\, \partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (614)$$

or

$$\frac{\partial^2 \phi}{\partial x\, \partial t} = -\rho \frac{\partial g}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (615)$$

From equation 612

$$\frac{\partial^2 \phi}{\partial t\, \partial x} = -4\pi\mu \int_0^x \frac{\partial g}{\partial t} dx \quad . \quad . \quad . \quad . \quad (616)$$

From equations 615 and 616

$$\rho \frac{\partial g}{\partial x} = 4\pi\mu \int_0^x \frac{\partial g}{\partial t} dx$$

Differentiating this with respect to x gives

$$\rho\frac{\partial^2 g}{\partial x^2} = 4\pi\mu\frac{\partial g}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (617)$$

Assuming that g varies sinoidally and interpreting equation 617 as a vector equation one obtains

$$\frac{d^2 g}{dx^2} - j\frac{4\pi\mu\omega}{\rho}g = 0 \quad . \quad . \quad . \quad . \quad . \quad (618)$$

the solution to which is

$$g = A_1 e^{k\sqrt{j}x} + A_2 e^{-k\sqrt{j}x} \text{ where } k^2 = \frac{4\pi\mu\omega}{\rho}$$

Since g is the same for $\pm x$, $A_1 = A_2$ and the solution may be written

$$g = A(e^{k\sqrt{j}x} + e^{-k\sqrt{j}x}) \quad . \quad . \quad . \quad . \quad (619)$$

or

$$g = 2A\cosh k\sqrt{j}\,x \quad . \quad . \quad . \quad . \quad . \quad (620)$$

The current per centimetre width of the conductor is

$$I_1 = \frac{2I_x}{b} = 2\int_0^a g\,dx$$

or

$$I_1 = 4A\frac{\sinh k\sqrt{j}a}{k\sqrt{j}} \quad . \quad . \quad . \quad . \quad . \quad (621)$$

if $\varepsilon' = g\rho + \dfrac{\partial\phi}{\partial t}$ then at the surface

$$\varepsilon' = g_0\rho = 2A\rho\cosh k\sqrt{j}a \quad . \quad . \quad . \quad (622)$$

and therefore the impedance per centimetre of 1 cm. width of the conductor is

$$Z_1 = \frac{\rho k\sqrt{j}}{2}\frac{\cosh k\sqrt{j}a}{\sinh k\sqrt{j}a} \quad . \quad . \quad . \quad . \quad (623)$$

(This impedance does not include the " external " reactance ; this does not matter since Z_1 is being determined to find R_1, the resistance of 1 cm. width of the strip.)

Writing, as before, $\alpha^2 = \dfrac{k^2}{2}$ then $k\sqrt{j} = \alpha(1+j)$ and

$$Z_1 = \frac{\rho(1+j)\alpha}{2}\left[\frac{\cosh\alpha a\cos\alpha a + j\sinh\alpha a\sin\alpha a}{\sinh\alpha a\cos\alpha a + j\cosh\alpha a\sin\alpha a}\right] \quad (624)$$

Writing $C = \cosh \alpha a$, $c = \cos \alpha a$, $S = \sinh \alpha a$, $s = \sin \alpha a$ one obtains

$$Z_1 = \frac{\rho\alpha(1+j)}{2}\left[\frac{CS - jcs}{S^2 + s^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (625)$$

giving $R_1 = \frac{\rho\alpha}{2}\left(\frac{CS + cs}{S^2 + s^2}\right) = \frac{\rho\alpha}{2}\left(\frac{\sinh 2\alpha a + \sin 2\alpha a}{\cosh 2\alpha a - \cos 2\alpha a}\right)$

putting $2a = t$, the thickness of the strip gives

$$R_1 = \frac{\rho\alpha}{2} \cdot \frac{\sinh \alpha t + \sin \alpha t}{\cosh \alpha t - \cos \alpha t} \quad . \quad . \quad . \quad (626)$$

Resistance of Strip at Power Frequencies.

Equation 626 may be used to determine the increase in resistance of strip conductors due to eddy currents at power frequencies.

To the first order, when α is small,

$$R_1 = \frac{\rho\alpha}{2} \cdot \frac{\alpha t + \alpha t}{1 + \frac{\alpha^2 t^2}{2} - 1 + \frac{\alpha^2 t^2}{2}} = \frac{\rho}{t} \text{ the D.C. value}$$

$$\frac{\text{A.C. resistance}}{\text{D.C. resistance}} = \frac{R_f}{R_0} = \frac{\alpha t}{2} \cdot \frac{\sinh \alpha t + \sin \alpha t}{\cosh \alpha t - \cos \alpha t} = \lambda \quad (627)$$

For copper $k = 0{\cdot}215\sqrt{f}$ and therefore $\alpha = 0{\cdot}152\sqrt{f}$. At $f = 50$, $\alpha = 1{\cdot}075$. For a strip 1 cm. thick $\alpha t = 1{\cdot}075$; $\sinh 1{\cdot}075 = 1{\cdot}2943$; $\sin 1{\cdot}075 = 0{\cdot}8796$; $\cosh 1{\cdot}075 = 1{\cdot}6356$; $\cos 1{\cdot}075 = 0{\cdot}4757$; and so

$$\lambda = \frac{1{\cdot}075}{2} \times \frac{2{\cdot}1739}{1{\cdot}1599} = 1{\cdot}0074$$

an increase of resistance of 0·74 per cent.

An approximate expression for λ, when αt is small, is

$$\lambda = 1 + \frac{(\alpha t)^4}{180} - \frac{(\alpha t)^8}{75{,}600} \quad . \quad . \quad . \quad . \quad . \quad . \quad (628)$$

High-frequency Resistance of Strip.

At very high frequencies, when αt becomes big, $\sinh \alpha t$ and $\cosh \alpha t$ become equally very great, while $\sin \alpha t$ and $\cos \alpha t$ remain finite so that the resistance of unit width of strip approximates asymptotically to $\frac{\rho\alpha}{2}$. Thus at $f = 10^6$, $\alpha = 152$ and a strip of considerable thickness would have a resistance of 76ρ, independently of the actual value of the thickness. It is clear,

however, from equation 626, that at a constant frequency, as the thickness is increased, the resistance passes through a number of maxima and minima. A strip of finite thickness may have a lower resistance than one which is indefinitely thick. Writing $S = \sinh \alpha t$, $s = \sin \alpha t$, $C = \cosh \alpha t$, $c = \cos \alpha t$, then

if $$y = \frac{S + s}{C - c}$$

$$\frac{dy}{d(\alpha t)} = \frac{(C + c)(C - c) - (S + s)(S + s)}{(C - c)^2}$$

Maximum and minimum values occur when

$$C^2 - c^2 - (S + s)^2 = 0$$

$$C^2 - c^2 - S^2 - 2Ss - s^2 = 0$$

or $$Ss = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (629)$$

Since both S and s being zero corresponds to a conductor of no thickness this case cannot be considered. $S \neq 0$, therefore $s = 0$ for a maximum or a minimum value.

At maximum or minimum values $\alpha t = \pi$, 2π, 3π, 4π, etc. When $\alpha t = 2n\pi$, $\cos \alpha t = +1$ and

$$\frac{S + s}{C - c} = \frac{\sinh \alpha t}{\cosh \alpha t - 1} = \frac{2 \sinh \frac{\alpha t}{2} \cosh \frac{\alpha t}{2}}{2 \sinh^2 \frac{\alpha t}{2}} = \coth \frac{\alpha t}{2}$$

$$= \coth 2n \cdot \frac{\pi}{2}$$

When $\alpha t = (2n + 1)\pi$, $\cos \alpha t = -1$

$$\frac{S + s}{C - c} = \frac{\sinh \alpha t}{\cosh \alpha t + 1} = \frac{2 \sinh \frac{\alpha t}{2} \cosh \frac{\alpha t}{2}}{2 \cosh^2 \frac{\alpha t}{2}} = \tanh \frac{\alpha t}{2}$$

$$= \tanh (2n + 1)\frac{\pi}{2}$$

Successive maximum values of R_1 are

$$\frac{\rho\alpha}{2}[\coth \pi, \coth 2\pi, \coth 3\pi \ . \ . \ .]$$

Successive minimum values of R_1 are

$$\frac{\rho\alpha}{2}\left[\tanh \frac{\pi}{2}, \tanh \frac{3\pi}{2}, \tanh \frac{5\pi}{2} \ . \ . \ .\right]$$

The particular value of interest is $\tanh \frac{\pi}{2} = 0{\cdot}917$; all sub-

sequent values differ from unity by only a negligible amount. For this case, when $\alpha t = \pi$, i.e. $t = \pi/152 = 0{\cdot}0206$ cm., the resistance is 8·3 per cent. less than the resistance of a thick strip.

Penetration of Flux into a Lamination at High Frequencies.

Fig. 87 (page 259) may now represent the cross-section of a lamination (the thickness greatly exaggerated). The magnetic force is perpendicular to the plane of the paper. Eddy e.m.f.s are generated along paths such as $ABCD$, giving rise to eddy currents along these paths. The total flux enclosed in $ABCD$ is ϕ; at a distance x from the central plane OO' the magnetic force is H, then

$$\phi = 2b\int_0^x \mu H \, dx \quad . \quad . \quad . \quad . \quad . \quad (630)$$

Variation of ϕ causes an e.m.f. to be generated along the path $ABCD$ equal to

$$\varepsilon = -\frac{\partial \phi}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (631)$$

and in consequence there are eddy currents of intensity

$$g = \frac{\varepsilon}{2b\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (632)$$

Passing outwards from x to $x + \delta x$ (and from $-x$ to $-x - \delta x$) the encircling current per unit length perpendicular to the paper diminishes by $g\,\delta x$ and the change in the magnetic force δH is 4π times this or

$$\frac{\partial H}{\partial x} = -4\pi g \quad . \quad . \quad . \quad . \quad . \quad (633)$$

From equations 630 and 631

$$\varepsilon = -2b\int_0^x \mu \frac{\partial H}{\partial t} dx \quad . \quad . \quad . \quad . \quad (634)$$

or from equation 632

$$g = -\frac{\mu}{\rho}\int_0^x \frac{\partial H}{\partial t} dx \quad . \quad . \quad . \quad . \quad (635)$$

Substituting from equation 633

$$\frac{\partial H}{\partial x} = \frac{4\pi\mu}{\rho}\int_0^x \frac{\partial H}{\partial t} dx \quad . \quad . \quad . \quad . \quad (636)$$

Differentiating with respect to x gives

$$\frac{\partial^2 H}{\partial x^2} = \frac{4\pi\mu}{\rho}\frac{\partial H}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (637)$$

Assuming that H varies sinoidally and interpreting equation 637 as a vector equation one obtains

$$\frac{d^2H}{dx^2} - j\frac{4\pi\mu\omega}{\rho}H = 0 \quad . \quad . \quad . \quad . \quad (638)$$

the applicable solution to which is

$$H = 2A \cosh k\sqrt{j}\, x \quad . \quad . \quad . \quad . \quad (639)$$

Following the work subsequent to equation 618 it is seen that the total flux per centimetre width of the lamination is

$$\phi_1 = \frac{4\mu A \sinh k\sqrt{j}a}{k\sqrt{j}} \quad . \quad . \quad . \quad . \quad (640)$$

Writing, as before, $\frac{k^2}{2} = \alpha^2$

$$H_0 = 2A \cosh (1 + j)\alpha a \quad . \quad . \quad . \quad . \quad (641)$$

and

$$\phi_1 = \frac{4\mu A \sinh (1 + j)\alpha a}{\alpha(1 + j)} \quad . \quad . \quad . \quad . \quad (642)$$

and putting $\cosh \alpha a = C$, $\sinh \alpha a = S$, $\cos \alpha a = c$, $\sin \alpha a = s$

$$H_0 = 2A(Cc + jSs)$$

$$\phi_1 = \frac{4\mu A}{\alpha(1 + j)}(Sc + jCs)$$

giving

$$|\phi_1| = \frac{4\mu A}{\alpha\sqrt{2}}\sqrt{S^2 + s^2}$$

and

$$|H_0| = 2A\sqrt{S^2 + c^2}$$

whence the value of the effective permeability is

$$\mu_e = \frac{\mu}{\alpha a\sqrt{2}}\sqrt{\frac{S^2 + s^2}{S^2 + c^2}} \quad . \quad . \quad . \quad . \quad (643)$$

When αa is very small

$$S^2 + s^2 = 2(\alpha a)^2$$

$$S^2 + c^2 = 1$$

giving

$$\mu_e = \mu$$

For iron if $\mu = 10^3$, and $\rho = 10^4$, $k = 2{\cdot}81\sqrt{f}$, $\alpha = 1{\cdot}985\sqrt{f}$. At $f = 50$, $\alpha = 14{\cdot}05$. If $a = 0{\cdot}02$, $a\alpha = 0{\cdot}281$, then $S = 0{\cdot}2847$, $s = 0{\cdot}2773$, $c = 0{\cdot}9608$ and $\mu_e = 1000$ to the nearest whole number.

When $a\alpha = 1$, corresponding to $\alpha = 50$ or $f = 635$, $\mu_e = 809$.

When $a\alpha = 3$, corresponding to $\alpha = 150$ or $f = 5720$, $\mu_e = 233$.

CHAPTER XIX

SOME SIMPLE PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATION

Partial differential equations involve the derivatives of one or more dependent variables with respect to two or more independent variables. Thus one may have equations connecting the current and voltage on a transmission line with the position along the line and the time. Here the independent variables are the distance x and the time t, upon which the current and the voltage, i and v, depend.

The general theory of partial differential equations is very difficult, but fortunately in most practical cases the general form of the solution is known from physical considerations. The actual solution is then much simplified.

Before proceeding to consider the solution of an actual equation it is useful to obtain an equation from its solution. For instance if

$$y = f(x) + \phi(t) \quad . \quad . \quad . \quad . \quad . \quad (644)$$

then, differentiating with respect to x, one obtains

$$\frac{\partial y}{\partial x} = f'(x) + 0 \quad . \quad . \quad . \quad . \quad . \quad (645)$$

Then differentiating with respect to t, the equation becomes

$$\frac{\partial^2 y}{\partial t \, \partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (646)$$

The process may now be reversed. Were equation 646 an ordinary differential equation one would, on integrating, add an arbitrary constant. Here, considering integration with respect to t, the addition of an arbitrary constant is not sufficient for a general solution. An arbitrary function of x must be added instead ; this vanishes on differentiation with respect to t. And so the integration of equation 646 with respect to t gives

$$\frac{\partial y}{\partial x} = F(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (647)$$

The same argument applies on integration with respect to x and one obtains as the solution

$$y = \int F(x)\,dx + \psi(t) \quad . \quad . \quad . \quad . \quad (647a)$$

It is at once apparent that $\int F(x)dx$ is a completely arbitrary function of x and $\psi(t)$ is a completely arbitrary function of t. The variety of solutions possible to such an equation as 646 is infinite. Difficulty may arise in practice, not merely in the analytical work of solution, but in selection of the solution applicable in the physical circumstances.

It is not proposed in this chapter to consider the general theory but to confine attention to certain partial differential equations which occur in practice and to discuss their solution.

An important solution merits attention, namely

$$y = f(x - at) \quad . \quad . \quad . \quad . \quad . \quad (648)$$

This expresses the dependence of some variable y upon two independent variables x and t. To fix attention, x may be imagined to be the distance measured from some fixed point along the air gap of an induction motor. t may represent time and y the flux density at the point defined by x. At a time t_1 the flux density at every point is defined by $y = f(x - at_1)$. At some arbitrary point x_1, the flux density $y_1 = f(x_1 - at_1)$. At a later time t_2 the flux density at a point x_2 so chosen that $x_2 - x_1 = a(t_2 - t_1)$ is

$$y = f(x_2 - at_2) = f[x_1 + a(t_2 - t_1) - at_2] = f(x_1 - at_1) = y_1.$$

That is to say, the state of affairs existing at x_1 at time t_1 is reproduced at x_2 at a time t_2. As x_1 has been arbitrarily chosen this applies to every point in the flux distribution. The flux distribution has moved forward a distance $x_2 - x_1$ in a time $t_2 - t_1$. But $(x_2 - x_1)/(t_2 - t_1) = a$. That is to say that equation 648 indicates that the particular distribution of flux represented by f is moving along the air gap with a velocity a. The nature of the actual distribution is not defined. Were it the ideal sinoidal distribution, equation 648 would be $y = A \sin b(x - at)$. An equation of the form of 648 is so common that its general differential counterpart must be con-

sidered. Differentiating equation 648 with respect to x gives

$$\frac{\partial y}{\partial x} = f'(x - at)$$

and

$$\frac{\partial^2 y}{\partial x^2} = f''(x - at) \quad . \quad . \quad . \quad . \quad . \quad (649)$$

Differentiating equation 648 with respect to t gives

$$\frac{\partial y}{\partial t} = -af'(x - at)$$

and

$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x - at) \quad . \quad . \quad . \quad . \quad (650)$$

giving

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (651)$$

Here the arbitrary function has been eliminated. Evidently equation 651 would also be obtained if $y = F(x + at)$, a disturbance moving in the opposite direction with the same speed, and

$$y = f(x - at) + F(x + at) \quad . \quad . \quad . \quad (652)$$

is a solution of equation 651. (This might represent two superposed flux distributions travelling in opposite directions.)

Given an equation of the form of 651 it may be known that equation 652 is the solution, and the nature of f and F may be fixed from physical considerations. An attempt at direct solution may be made. Equation 651 is linear and from previous experience it is known that ordinary linear equations have solutions of exponential form. Suppose it be assumed that a solution is possible of the form

$$y = e^{mx+nt} \quad . \quad . \quad . \quad . \quad . \quad (653)$$

then

$$\frac{\partial^2 y}{\partial t^2} = n^2 e^{mx+nt}$$

and

$$\frac{\partial^2 y}{\partial x^2} = m^2 e^{mx+nt}$$

Equation 651 is satisfied if $n^2 = a^2 m^2$ or $n = \pm am$. A possible solution is therefore

$$y = Ae^{mx+amt} + Be^{mx-amt} \quad . \quad . \quad . \quad (654)$$

This is a particular form of equation 652. Now it may be known that y diminishes as x increases, becoming vanishingly small when x is very great. In this case it is evident that A must be zero if m is positive and the applicable solution is

$$y = Be^{mx-amt} \quad . \quad . \quad . \quad . \quad . \quad (655)$$

Another important equation is

$$\frac{\partial^2 y}{\partial x^2} = -a^2 y \quad . \quad . \quad . \quad . \quad . \quad (656)$$

where y is a function not only of x but also of t. Were this an ordinary differential equation the solution would be

$$y = A \cos ax + B \sin ax$$

This solution is extended to the partial differential equation by writing arbitrary functions of t for the constants of integration, giving

$$y = f(t) \cos ax + F(t) \sin ax \quad . \quad . \quad . \quad (657)$$

Electromagnetic Radiation.

Consider any point in an electromagnetic field the medium being insulating and having a permeability μ and dielectric constant κ. Two fundamental relations have been established, namely

(i) the line integral of the magnetic force H round any closed path is equal to $\frac{4\pi}{c} \cdot \frac{\partial N}{\partial t}$ where N is the electric flux linking the path (equation 97, p. 71).

(ii) the line integral of the electric force ε round any closed path is equal to $-\frac{1}{c}\frac{\partial \phi}{\partial t}$ where ϕ is the magnetic flux linking the path (equation, 99 p. 72). Consider (i). If the path enclose a very small area δS then $N = D_n \delta S$ where D_n is the component displacement, or electric flux density, normal to the area. But $D_n = \frac{\kappa \varepsilon_n}{4\pi}$ (equation 53, p. 46) and so the line integral round the path

$$\oint H \cos\theta \, ds = \frac{\kappa}{c}\frac{\partial \varepsilon_n}{\partial t} \delta S$$

or

$$\frac{\oint H \cos\theta \, ds}{\delta S} = \frac{\kappa}{c}\frac{\partial \varepsilon_n}{\partial t}$$

But the left-hand member is the definition of the curl of H (equation 84, p. 64)—the component of course in this case perpendicular to the plane of the area, or

$$\text{Curl } H = \frac{\kappa}{c}\frac{\partial \varepsilon}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (658)$$

Consider (ii). If the path enclose a very small area δS then $\phi = \mu H_n \delta S$ and

$$\oint \varepsilon \cos\theta \, ds = -\frac{\mu}{c}\frac{\partial H_n}{\partial t}\delta S$$

or

$$\text{Curl } \varepsilon = -\frac{\mu}{c}\frac{\partial H}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (659)$$

The components of ε along the axes of x, y, and z are X, Y, Z; if the components of H along the axes of x, y and z are α, β, γ then, expressing in Cartesian co-ordinates, equation 658 becomes

$$\left.\begin{aligned} \frac{\kappa}{c}\frac{\partial X}{\partial t} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \\ \frac{\kappa}{c}\frac{\partial Y}{\partial t} &= \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \\ \frac{\kappa}{c}\frac{\partial Z}{\partial t} &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (660)$$

(see expressions for curl on p. 66)

and equation 659 becomes

$$\left.\begin{aligned} -\frac{\mu}{c}\frac{\partial \alpha}{\partial t} &= \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \\ -\frac{\mu}{c}\frac{\partial \beta}{\partial t} &= \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \\ -\frac{\mu}{c}\frac{\partial \gamma}{\partial t} &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (661)$$

Differentiating the first of equation 660 with respect to t gives

$$\frac{\kappa}{c}\frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 \gamma}{\partial y \, \partial t} - \frac{\partial^2 \beta}{\partial z \, \partial t}$$

Substituting from equation 661 one obtains

$$\frac{\kappa}{c}\frac{\partial^2 X}{\partial t^2} = -\frac{c}{\mu}\left[\frac{\partial^2 Y}{\partial x \, \partial y} - \frac{\partial^2 X}{\partial y^2} - \frac{\partial^2 X}{\partial z^2} + \frac{\partial^2 Z}{\partial x \, \partial z}\right] . \quad (662)$$

Since there is no space charge, Gauss's theorem (equation 46, p. 44) gives

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (663)$$

or

$$\frac{\partial^2 X}{\partial x^2} = -\frac{\partial^2 Y}{\partial x\,\partial y} - \frac{\partial^2 Z}{\partial x\,\partial z} \quad . \quad . \quad . \quad . \quad (664)$$

Substituting equation 664 in equation 662 gives

$$\frac{\partial^2 X}{\partial t^2} = \frac{c^2}{\mu\kappa}\left(\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2}\right) \quad . \quad . \quad . \quad (665)$$

Similar equations are obtained for Y and Z and also for α, β, γ.

Particular Solution—Plane Wave.

Equation 665 is the three-dimensional analogue of equation 651 and represents a wave travelling (or two oppositely directed waves) with a velocity $\frac{c}{\sqrt{\mu\kappa}}$.

To investigate the equation further, a simple case is considered in which the electric and magnetic forces are wholly in a plane perpendicular to the axis of x. Throughout this plane the electric and magnetic forces are constant at any instant. X is zero, but the equation in Y corresponding to 665 becomes

$$\frac{\partial^2 Y}{\partial t^2} = \frac{c^2}{\mu\kappa}\frac{\partial^2 Y}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (666)$$

giving

$$Y = f\left(x - \frac{c}{\sqrt{\mu\kappa}}t\right) + F\left(x + \frac{c}{\sqrt{\mu\kappa}}t\right) \quad . \quad . \quad (667)$$

and similar equations in Z, β and γ. Equation 667 is one aspect of two disturbances travelling along the axis of x with a velocity $\frac{c}{\sqrt{\mu\kappa}}$. The direction of motion is perpendicular to the plane containing the electric and magnetic forces.

Generally, equations 660 and 661 degenerate to

$$\frac{\kappa}{c}\frac{\partial X}{\partial t} = 0\,; \quad \frac{\kappa}{c}\frac{\partial Y}{\partial t} = -\frac{\partial \gamma}{\partial x}\,; \quad \frac{\kappa}{c}\frac{\partial Z}{\partial t} = \frac{\partial \beta}{\partial x} \quad . \quad (668)$$

$$-\frac{\mu}{c}\frac{\partial \alpha}{\partial t} = 0\,; \quad -\frac{\mu}{c}\frac{\partial \beta}{\partial t} = -\frac{\partial Z}{\partial x}\,; \quad -\frac{\mu}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial Y}{\partial x} \quad (669)$$

Since $\frac{\partial X}{\partial t}$ and $\frac{\partial \alpha}{\partial t}$ are both zero X and α are zero or constant in time and constitute no part of the wave motion. *The electric and magnetic forces are transverse to the direction of travel of the wave.* The axes may be considered to be rotated about the axis of x so that the axis of y coincides with the direction of the electric force. Then $Z = 0$ and $\frac{\partial \beta}{\partial x} = 0$. The only component of magnetic force remaining is therefore γ, coinciding with the axis of z. That is to say, *the electric force, the magnetic force, and the direction of motion are mutually perpendicular.* The remaining equations are

$$\frac{\kappa}{c}\frac{\partial Y}{\partial t} = -\frac{\partial \gamma}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (670)$$

$$-\frac{\mu}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial Y}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (671)$$

giving

$$\frac{\partial^2 Y}{\partial t^2} = \frac{c^2}{\mu\kappa}\frac{\partial^2 Y}{\partial x^2} \quad . \quad . \quad . \quad . \quad (672)$$

and

$$\frac{\partial^2 \gamma}{\partial t^2} = \frac{c^2}{\mu\kappa}\frac{\partial^2 \gamma}{\partial x^2} \quad . \quad . \quad . \quad . \quad (673)$$

equations identical in form with 651.

Particular Case—Spherical Wave.

To consider this case it is convenient to change to spherical polar co-ordinates (Fig. 88) $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$. Then it can be shown (the work is straightforward but rather laborious and is not given here) that

$$\nabla^2\alpha = \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2}$$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\alpha}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\alpha}{\partial\phi^2}\right] \quad (674)$$

If the solution is to be symmetrical about the origin, that is independent of ϕ and θ, equation 665 becomes

$$\frac{\partial^2 X}{\partial t^2} = \frac{c^2}{\mu\kappa}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial X}{\partial r}\right) \quad . \quad . \quad . \quad . \quad (675)$$

putting $W = rX$

$$\frac{\partial W}{\partial r} = X + r\frac{\partial X}{\partial r}$$

$$\frac{\partial^2 W}{\partial r^2} = r\frac{\partial^2 X}{\partial r^2} + 2\frac{\partial X}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial X}{\partial r}\right)$$

Substituting in equation 675 gives

$$\frac{\partial^2 W}{\partial t^2} = \frac{c^2}{\mu\kappa}\frac{\partial^2 W}{\partial r^2} \quad . \quad . \quad . \quad . \quad . \quad (676)$$

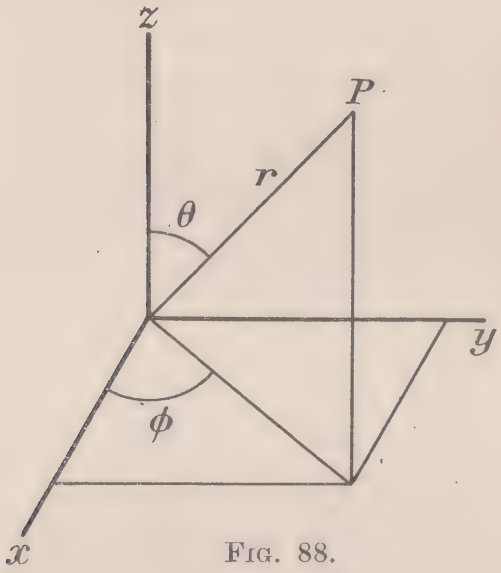


FIG. 88.

of which the solution is

$$W = f\left(r - \frac{c}{\sqrt{\mu\kappa}}t\right) + F\left(r + \frac{c}{\sqrt{\mu\kappa}}t\right)$$

or

$$X = \frac{1}{r}f\left(r - \frac{c}{\sqrt{\mu\kappa}}t\right) + \frac{1}{r}F\left(r + \frac{c}{\sqrt{\mu\kappa}}t\right) \quad . \quad (677)$$

These are two radial waves of velocity $\frac{c}{\sqrt{\mu\kappa}}$ the intensity being inversely proportional to the distance from the origin.

Transmission Along Wires.

Per unit of length the line has a resistance R, inductance L, capacitance C and leakance G. Then (Fig. 89)

$$-\frac{\partial v}{\partial x}\delta x = iR\ \delta x + L\frac{\partial i}{\partial t}\delta x$$

$$-\frac{\partial i}{\partial x}\delta x = vG\ \delta x + C\frac{\partial v}{\partial t}\delta x$$

or
$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (678)$$

$$-\frac{\partial i}{\partial x} = Gv + C\frac{\partial v}{\partial t} \quad . \quad . \quad . \quad . \quad (679)$$

Differentiating equation 678 with respect to x gives

$$-\frac{\partial^2 v}{\partial x^2} = R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x\,\partial t} \quad . \quad . \quad . \quad . \quad (680)$$

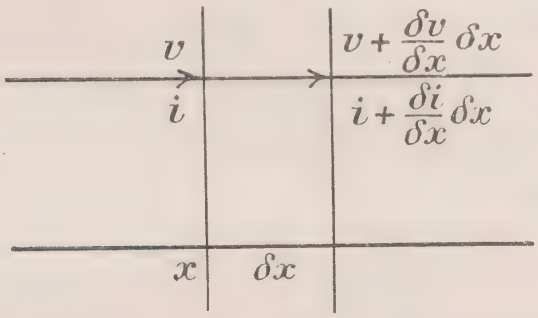


Fig. 89.

Differentiating equation 679 with respect to t gives

$$-\frac{\partial^2 i}{\partial t\,\partial x} = G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2} \quad . \quad . \quad . \quad . \quad (681)$$

Substituting equations 681 and 679 in equation 680 gives

$$CL\frac{\partial^2 v}{\partial t^2} + (CR + GL)\frac{\partial v}{\partial t} + GRv = \frac{\partial^2 v}{\partial x^2} \quad . \quad . \quad (682)$$

and similarly

$$CL\frac{\partial^2 i}{\partial t^2} + (CR + GL)\frac{\partial i}{\partial t} + GRi = \frac{\partial^2 i}{\partial x^2} \quad . \quad . \quad (683)$$

The solution of these equations will be considered under certain particular circumstances.

Steady Sinoidal Conditions.

Assuming steady sinoidal conditions, V and I being the vector voltage and vector current respectively then equations 682 and 683 become

$$-\omega^2 CLV + j\omega(CR + GL)V + GRV = \frac{d^2 V}{dx^2}$$

and
$$-\omega^2 CLI + j\omega(CR + GL)I + GRI = \frac{d^2 I}{dx^2}$$

or
$$\frac{d^2V}{dx^2} = (R + j\omega L)(G + j\omega C)V \quad . \quad . \quad . \quad (684)$$

and
$$\frac{d^2I}{dx^2} = (R + j\omega L)(G + j\omega C)I \quad . \quad . \quad . \quad (685)$$

Putting $P^2 = (R + j\omega L)(G + j\omega C)$ the solution of equation 684 may be written
$$V = A \cosh Px + B \sinh Px \quad . \quad . \quad . \quad (686)$$

and the solution of equation 685 as
$$I = A' \cosh Px + B' \sinh Px \quad . \quad . \quad . \quad (687)$$

There are of course relations between A, B and A', B'. Writing equation 679 as
$$-\frac{dI}{dx} = (G + j\omega C)V$$

then from equation 686
$$-\frac{dI}{dx} = (G + j\omega C)A \cosh Px + (G + j\omega C)B \sinh Px$$

but from equation 687
$$-\frac{dI}{dx} = -PA' \sinh Px - PB' \cosh Px$$

whence
$$A' = -B\sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

and
$$B' = -A\sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

$\sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$ is of the dimensions of an impedance. Without, for the moment, any consideration of its physical meaning, this will be written as Z_0. Then
$$A' = -\frac{B}{Z_0} \text{ and } B' = -\frac{A}{Z_0}$$

The equations then become
$$V = A \cosh Px + B \sinh Px \quad . \quad . \quad . \quad (688)$$

and
$$I = -\frac{B}{Z_0} \cosh Px - \frac{A}{Z_0} \sinh Px \quad . \quad . \quad . \quad (689)$$

These equations are not confined to the particular nature of the line impedance and line admittance assumed. If, more generally, one writes Z = impedance per unit length of line and Y = admittance per unit length of line then

$$P = \sqrt{ZY} \text{ and } Z_0 = \sqrt{Z/Y} \quad . \quad . \quad . \quad (690)$$

If the voltage and current at the sending end of the line ($x = 0$) are V_0 and I_0 respectively then

$$V_0 = A \text{ and } I_0 = -\frac{B}{Z_0} \text{ giving}$$

$$V = V_0 \cosh Px - I_0 Z_0 \sinh Px \quad . \quad . \quad (691)$$

and

$$I = I_0 \cosh Px - \frac{V_0}{Z_0} \sinh Px \quad . \quad . \quad . \quad (692)$$

The physical interpretation of this equation is made more clear by substituting for P the sum of its real and imaginary parts

$$P = \alpha + j\beta$$

Equating gives

$$(\alpha + j\beta)^2 = (R + j\omega L)(G + j\omega C)$$

$$\alpha^2 - \beta^2 = RG - \omega^2 LC$$

$$2\alpha\beta = \omega(LG + CR)$$

$$\alpha^2 + \beta^2 = \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}$$

$$= \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}$$

giving

$$\alpha^2 = \tfrac{1}{2}[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + (RG - \omega^2 LC)] \quad . \quad (693)$$

$$\beta^2 = \tfrac{1}{2}[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} - (RG - \omega^2 LC)] \quad . \quad (694)$$

Then equations 691 and 692 may be written as

$$\left.\begin{aligned} V &= \tfrac{1}{2}(V_0 - I_0 Z_0)\varepsilon^{Px} + \tfrac{1}{2}(V_0 + I_0 Z_0)\varepsilon^{-Px} \\ \text{and} \quad I &= \tfrac{1}{2}(I_0 - V_0/Z_0)\varepsilon^{Px} + \tfrac{1}{2}(I_0 + V_0/Z_0)\varepsilon^{-Px} \end{aligned}\right\}$$

or

$$V = \tfrac{1}{2}(V_0 - I_0 Z_0)e^{\alpha x}(\cos \beta x + j \sin \beta x) + \tfrac{1}{2}(V_0 + I_0 Z_0)e^{-\alpha x}(\cos \beta x - j \sin \beta x) \quad . (695)$$

$$I = \tfrac{1}{2}(I_0 - V_0/Z_0)e^{\alpha x}(\cos \beta x + j \sin \beta x) + \tfrac{1}{2}(I_0 + V_0/Z_0)e^{-\alpha x}(\cos \beta x - j \sin \beta x) \quad . \quad (696)$$

The physical meaning of these expressions can be gathered by examining one of them, say that for V, and, for the moment,

only the second half of the expression. $\frac{1}{2}(V_0 + I_0Z_0)$ is a voltage vector; $e^{-\alpha x}$ is a numerical multiplier, which decreases the voltage in the ratio $e^{-\alpha}$ per unit length of line; $(\cos \beta x - j \sin \beta x)$ is a vector operator which retards the phase of the voltage vector by an angle β per unit distance. Exactly this state of affairs could be brought about by a wave travelling along the line in the direction of power propagation at a speed ω/β, the wave diminishing in the ratio $e^{-\alpha}$ per unit distance travelled. The wave-length is thus defined by β, and the attenuation by α. The first term in the expression for V represents an exactly similar wave travelling in the opposite direction at the same speed, decreasing and changing in phase in the same manner. It is thus seen that the voltage at any point is consistent with the view that two pressure-waves are travelling along it, attenuating as they travel. The second term in the expression for V is the resultant of an infinitely large number of forward wave trains; the first is the resultant of all reflected wave trains.* The expression for I is made up in an exactly similar manner.

Infinitely Long Line.

If the line is infinitely long then the current must be zero at infinite distance from the transmitting end. The first factor, $\frac{1}{2}(I_0 - V_0/Z_0)$, in the expression for I must be zero and hence $I_0 = V_0/Z_0$ and equations 695 and 696 become

$$V = V_0e^{-\alpha x}(\cos \beta x - j \sin \beta x)$$
$$I = I_0e^{-\alpha x}(\cos \beta x - j \sin \beta x)$$

The effective impedance of the line is Z_0. There is no reflected wave. Z_0 is called the *characteristic impedance*; P is the *propagation constant*; α is called the *attenuation constant*; β is called the *wave-length constant*.

Distortionless Line.

A case of interest arises if $LG = RC$. Writing this condition as $\frac{\omega L}{R} = \frac{\omega C}{G} = a$, one obtains

$$\alpha^2 = \tfrac{1}{2}\{GR(1 + a^2) + GR(1 - a^2)\} = GR$$

and

$$\alpha = \sqrt{GR} \quad . \quad . \quad . \quad . \quad . \quad . \quad (697)$$

* Warren, *Proc. Inst. Elect. Engs.*, Vol. 59, p. 330.

and $$\beta^2 = \tfrac{1}{2}\{GR(1 + a^2) - GR(1 - a^2)\} = a^2GR$$

$$\beta = a\sqrt{GR} = \omega\sqrt{LC} \quad . \quad . \quad . \quad . \quad (698)$$

The velocity of transmission is

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad . \quad . \quad . \quad . \quad . \quad (699)$$

For this line all frequencies are transmitted with the same velocity and are equally attenuated. The line is distortionless. It is not uncommonly said, though quite wrongly, that this is a line of minimum attenuation.

The relationship $LG = RC$ never holds in practice, as RC is always much greater than LG.

The expression for β may be more carefully examined

$$\begin{aligned}(\omega^2L^2+R^2)(\omega^2C^2+G^2) &= \omega^4L^2C^2+\omega^2L^2G^2+\omega^2C^2R^2+G^2R^2 \\ &= (\omega^2LC + GR)^2 + \omega^2L^2G^2 + \omega^2C^2R^2 - 2\omega^2LCGR \\ &= (\omega^2LC + GR)^2 + \omega^2(CR - LG)^2 \quad . \quad . \quad . \quad (700)\end{aligned}$$

And therefore

$$(\omega^2L^2 + R^2)(\omega^2C^2 + G^2) > (\omega^2LC + GR)^2$$

unless $CR = LG$ or R and G are both zero. It therefore follows that $\beta > \omega\sqrt{LC}$ unless $CR = LG$ or R and G are zero, and therefore the velocity of transmission $u \not> \dfrac{1}{\sqrt{LC}}$.

In the case of an open line $L = 4 \times 10^{-9}\mu \log \dfrac{d}{r}$ (equation 117, p. 83) and

$$C = \frac{1}{9 \times 10^{11}} \cdot \frac{\kappa}{4 \log \dfrac{d}{r}}$$

so that $LC = \dfrac{\mu\kappa}{9 \times 10^{20}}$ and $\dfrac{1}{\sqrt{LC}} = \dfrac{3 \times 10^{10}}{\sqrt{\mu\kappa}} = \dfrac{c}{\sqrt{\mu\kappa}}$

where c is the velocity of light in space.

In air $\mu = 1$ and $\kappa = 1$ and the limiting velocity is that of light and occurs in a line with no losses or if $CR = LG$. The product LC given above is an irreducible minimum (it may easily be increased), and the velocity of transmission cannot exceed the velocity of light.

Line with Small Losses.

The resistance of a line depends (at low frequencies at any rate) upon its cross-sectional area. Its reactance, ωL, depends upon its diameter and its spacing. At power frequencies the reactance is usually less than the resistance for the lighter lines but greater than the resistance for the heavier lines. For underground cables the reactance is generally less than the resistance. In any case the two quantities have a ratio which rarely exceeds 3 or 4 and is usually much less. At speech frequencies ωL is considerably greater than R, and ωC considerably greater than G. At carrier frequencies the disparity is still more marked and at radio frequencies $\omega L >> R$ and $\omega C >> G$. In this case the characteristic impedance becomes

$$Z_0 = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \doteqdot \sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad (701)$$

a pure resistance.

Under these conditions it is useful to determine a simple approximate expression for the attenuation constant α. From equation 700

$$\sqrt{(\omega^2 L^2 + R^2)(\omega^2 C^2 + G^2)} \doteqdot \omega^2 LC + GR + \frac{\omega^2 (CR - LG)^2}{2\omega^2 LC}$$

whence

$$\sqrt{(\omega^2 L^2 + R^2)(\omega^2 C^2 + G^2)} + (GR - \omega^2 LC)$$
$$= 2GR + \frac{C^2R^2 - 2CRLG + L^2G^2}{2LC}$$
$$= \frac{(CR + LG)^2}{2LC}$$

giving

$$\alpha \doteqdot \frac{CR + LG}{2\sqrt{LC}} = \frac{R}{2Z_0} + \frac{GZ_0}{2} \quad . \quad . \quad . \quad (702)$$

Generally the former of these is considerably greater than the latter, and the attenuation of the line may be reduced by increasing Z_0. This may be done by the addition of loading coils, as suggested by Heaviside, increasing L far more rapidly than R.

Impedance of Finite Terminated Line.

If V_0 and I_0 are respectively the voltage and current at the sending end of a line of length l, the voltage and current

at the receiving end, namely V_r and I_r are connected by the relations

$$V_r = V_0 \cosh Pl - I_0 Z_0 \sinh Pl \quad . \quad . \quad (703)$$

and

$$I_r = I_0 \cosh Pl - \frac{V_0}{Z_0} \sinh Pl \quad . \quad . \quad . \quad (704)$$

Viewing the line from the other end and writing $-l$ for l one must have the reciprocal relations

$$V_0 = V_r \cosh Pl + I_r Z_0 \sinh Pl \quad . \quad . \quad (705)$$

and

$$I_0 = I_r \cosh Pl + \frac{V_r}{Z_0} \sinh Pl \quad . \quad . \quad . \quad (706)$$

If the line is terminated by an impedance Z_r then

$$\frac{V_0}{I_0} = \frac{Z_r \cosh Pl + Z_0 \sinh Pl}{\cosh Pl + \frac{Z_r}{Z_0} \sinh Pl} \quad . \quad . \quad . \quad (707)$$

This is the *sending end impedance* of the line so terminated. If $Z_r = Z_0$, the characteristic impedance of the line, then $\frac{V_0}{I_0} = Z_0$ and the line behaves as if it were indefinitely long and there is no reflection. When a line has to transmit speech it is essential that it shall be non-reflective, to obviate echo effects. Also if it is not non-reflective the received intensity will be a function of the frequency and there will be considerable distortion. If, however, a single frequency (or group of frequencies with but a small percentage variation from the average) is being transmitted, as in carrier telephony or telegraphy, or in a radio feeder, efficiency only has to be considered. The greatest efficiency is obtained when the terminating impedance matches the line. The converse is not true.

If $Z_r = 0$, corresponding to a shortcircuited line, then

$$Z_g = \frac{V_0}{I_0} = Z_0 \tanh Pl \quad . \quad . \quad . \quad . \quad (708)$$

If $Z_r = \infty$, corresponding to an open line, then

$$Z_f = \frac{V_0}{I_0} = Z_0 \coth Pl \quad . \quad . \quad . \quad . \quad (709)$$

so that

$$Z_0 = \sqrt{Z_f Z_g} \quad . \quad . \quad . \quad . \quad . \quad (710)$$

(Z_g stands for the sending end impedance when the line is shortcircuited at the receiving end. Z_f corresponds when the receiving end of the line is open.)

Making these two measurements enables the characteristic impedance of the line to be determined.

In the case of a radio feeder $\omega L >> R$ and $\omega C >> G$. As a first approximation one may write $\alpha = 0$.

Then $\cosh Pl = \cosh j\beta l = \cos \beta l$

and $\sinh Pl = \sinh j\beta l = j \sin \beta l$

and the sending end impedance is

$$\frac{V_0}{I_0} = Z_0 \frac{Z_r \cos \beta l + jZ_0 \sin \beta l}{Z_0 \cos \beta l + jZ_r \sin \beta l} \quad . \quad . \quad . \quad (711)$$

If the line is one-quarter of a wave-length long $\beta l = \frac{\pi}{2}$ and the sending end impedance becomes

$$\frac{V_0}{I_0} = Z_0 \frac{jZ_0}{jZ_r} = \frac{Z_0^2}{Z_r} \quad . \quad . \quad . \quad . \quad (712)$$

If the line is shortcircuited ($Z_r = 0$) then its impedance is infinite ! If the line is open ($Z_r = \infty$) then its impedance is zero ! These lines are made use of in ultra high-frequency radio receivers. The shortcircuited line allows direct currents to pass freely, but is a very high impedance to the radio frequency currents. The open line is a stopper to direct currents but passes radio-frequency currents freely.

Another important application of the quarter wave-length line is the so-called *quarter wave-length transformer* used in short wave radio work. If the characteristic impedance of an aerial feeder is Z_1 and the impedance of the aerial is Z_r then the aerial may be matched to the feeder by the insertion of a quarter wave-length of line having a characteristic impedance Z_0 such that $Z_1 = \frac{Z_0^2}{Z_r}$, fulfilling equation 712. The required line is defined by

$$Z_0 = \sqrt{Z_1 Z_r} \quad . \quad . \quad . \quad . \quad . \quad (713)$$

The equation $\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}$.

This equation

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (714)$$

a simplified form of equation 682, is of great importance in both electrical theory and in the conduction of heat. Its comparative simplicity permits of a more adequate solution than was possible in the more general equation.

Suppose that a solution is required which shall fulfil the following conditions:

(i) v shall always be finite.

(ii) $v = 0$ at $x = 0$ and at $x = l$ for all values of t.

(iii) At $t = 0$, $v = 100$ at $x = \frac{l}{2}$ and it varies linearly from $x = 0$ to $x = \frac{l}{2}$ and from $x = \frac{l}{2}$ to $x = l$ (Fig. 90).

(a) If a uniform metal rod of length l has its ends in ice and its centre in boiling water, there being no loss of heat from the rod except by conduction along it, the conditions are represented by Fig. 90. If at $t = 0$, the boiling water be removed the temperature of the rod will fall, finally being reduced everywhere to 0° C. The differential equation expressing the changes of temperature which take place is equation 714 and the solution of this equation, subject to the conditions (i), (ii) and (iii) gives the temperature at any time t at any point x.

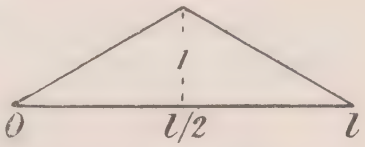


FIG. 90.

(b) If a uniform line of length l, having a resistance R and a capacitance C per unit length, L and G being negligible, has its two ends earthed and has its centre raised 100 volts above earth, by a battery connected between earth and the centre of the line, then the potential at various points along the line is represented by Fig. 90. If any change be made in the conditions, readjustments must be subject to the equations

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{CR}\frac{\partial^2 v}{\partial x^2} \\ \text{and} \quad \frac{\partial i}{\partial t} &= \frac{1}{CR}\frac{\partial^2 i}{\partial x^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (715)$$

(obtained by putting $L = 0$, $G = 0$ in equations 682 and 683).

These are both of the form of equation 714. The solution of this equation $\left(\text{with } k = \frac{1}{CR}\right)$ subject to the conditions (i),

(ii) and (iii) therefore gives the potential at any point along the line subsequent to the instant $t = 0$ at which the battery is disconnected.

It is assumed that the solution is of the form

$$v = e^{mx+nt}$$

Putting this in equation 714 gives $n = km^2$. Now if n were positive v would become infinite as t becomes infinite. Therefore n must be negative. Hence if k is positive, m^2 must be negative. If $m^2 = -p^2$ then $m = jp$ and $n = -kp^2$. A solution of the equation which fulfils condition (i) is therefore

$$v = Ae^{-kp^2t}\varepsilon^{jpx}$$

p may have any real value positive or negative so that another particular solution is

$$v = Ae^{-kp^2t}e^{-jpx}$$

and another $$v = Ae^{-kp^2t}(e^{jpx} - e^{-jpx})$$

or

$$v = Be^{-kp^2t} \sin px \quad . \quad . \quad . \quad . \quad . \quad (716)$$

If $pl = \pi$ or $p = \dfrac{\pi}{l}$ then equation 716 becomes

$$v = Be^{-k\frac{\pi^2}{l^2}t} \sin \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad (717)$$

Now this equation satisfies both conditions (i) and (ii), but it does not satisfy condition (iii). It would be the complete solution if condition (iii) required that at $t = 0$ the distribution of v along the line were represented by a sine curve between 0 and π. But equation 714 and conditions (i) and (ii) are equally well satisfied by p being any whole multiple, s, of $\dfrac{\pi}{l}$. And these conditions are equally well satisfied by the sum of a number of such solutions or

$$v = B_1e^{-k\frac{\pi^2}{l^2}t} \sin \frac{\pi x}{l} + B_2e^{-k\frac{2^2\pi^2}{l^2}t} \sin \frac{2\pi x}{l} + B_3e^{-k\frac{3^2\pi^2}{l^2}t} \sin \frac{3\pi x}{l} + \ldots$$

or

$$v = \sum_{s=1}^{\infty} B_se^{-k\frac{s^2\pi^2}{l^2}t} \sin \frac{s\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad (718)$$

Now it is shown in the next chapter that the function of v represented by Fig. 90 can be expressed in a Fourier series as

$$v = C\left(\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} \ldots\right) \quad (719)$$

where $C = \frac{8}{\pi^2}$ times the maximum value. In this case $C = \frac{800}{\pi^2}$. And therefore the complete solution satisfying equation 714 subject to the conditions (i), (ii) and (iii) is

$$v = \frac{800}{\pi^2}\left[e^{-k\frac{\pi^2}{l^2}t}\sin\frac{\pi x}{l} - \frac{1}{3^2}e^{-k\frac{3^2\pi^2}{l^2}t}\sin\frac{3\pi x}{l} + \frac{1}{5^2}e^{-k\frac{5^2\pi^2}{l^2}t}\sin\frac{5\pi x}{l} \ldots\right] \quad (720)$$

where $k = \frac{1}{CR}$.

The Telegraph Equation.

In the case of underground and submarine cables the important primary constants are the resistance and capacitance. No serious error is introduced by neglecting the inductance and leakance. Equations 678 and 679 then reduce to

$$\frac{\partial v}{\partial x} = -Ri \quad (721)$$

and

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \quad (722)$$

These give

$$\frac{\partial v}{\partial t} = \frac{1}{CR}\frac{\partial^2 v}{\partial x^2} \quad (723)$$

and

$$\frac{\partial i}{\partial t} = \frac{1}{CR}\frac{\partial^2 i}{\partial x^2} \quad (724)$$

Suppose that the sending end of this line is raised to a potential E, the far end being connected through a receiving instrument of negligible resistance to earth. After a steady state has been reached the voltage distribution along the line will be represented by

$$V = E\left(1 - \frac{x}{l}\right) \quad (725)$$

This condition having been established, the sending end of the line is connected to earth (at $t = 0$). The subsequent voltage relations along the line are represented by an equation of the form of 718. Now it is shown in the next chapter that the function of V represented by equation 725 can be expressed in a Fourier series as

$$v' = \frac{2E}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} \ldots\right]$$

or

$$v' = \frac{2E}{\pi}\sum_{s=1}^{\infty}\frac{1}{s}\sin\frac{s\pi x}{l} \quad \ldots\ldots \quad (726)$$

The relevant solution is therefore

$$v' = \frac{2E}{\pi}\sum_{s=1}^{\infty}\frac{1}{s}e^{-\frac{s^2\pi^2 t}{CRl^2}}\sin\frac{s\pi x}{l} \quad \ldots \quad (727)$$

It follows from equations 725 and 727 that if at $t = 0$ a voltage E is suddenly applied to the sending end of the line, the line being previously dead and earthed at its receiving end, that the subsequent potential at any point x is

$$v = V - v'$$

or

$$v = E\left(1 - \frac{x}{l}\right) - \frac{2E}{\pi}\sum_{s=1}^{\infty}\frac{1}{s}e^{-\frac{s^2\pi^2 t}{CRl^2}}\sin\frac{s\pi x}{l} \quad . \quad (728)$$

or as it is more commonly written

$$v = \frac{2E}{\pi}\left[\sum_{s=1}^{\infty}\frac{1}{s}\sin\frac{s\pi x}{l} - \sum_{s=1}^{\infty}\frac{1}{s}e^{-\frac{s^2\pi^2 t}{CRl^2}}\sin\frac{s\pi x}{l}\right] \quad (729)$$

The current is given by equation 721, $i = -\frac{1}{R}\frac{\partial v}{\partial x}$. Using equation 728 this gives

$$i = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}e^{-\frac{s^2\pi^2 t}{CRl^2}}\cos\frac{s\pi x}{l}\right] \quad . \quad (730)$$

At the sending end, $x = 0$ and

$$i_s = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}e^{-\frac{s^2\pi^2 t}{CRl^2}}\right] \quad \ldots \quad (731)$$

At the receiving end, $x = l$, and

$$i_r = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}(-1)^s e^{-\frac{s^2\pi^2 t}{CRl^2}}\right] \quad . \quad (732)$$

Arrival Curves.

Equation 732 may be written as

$$i_r = \frac{2E}{Rl}\left[\frac{1}{2} - e^{-ut} + e^{-4ut} - e^{-9ut} + e^{-16ut} \ldots\right] = \frac{2E}{Rl} f(ut)$$

where $u = \dfrac{\pi^2}{CRl^2}$ and $f(u, t) = \frac{1}{2} - e^{-ut} + e^{-4ut} - e^{-9ut} \ldots$

Now it can be proved that when $ut = 0$, $f(ut) = 0$ as would be expected from physical considerations. By giving values to ut the general form of the curve may be plotted. It is given in Fig. 91. The peculiarity of this curve is the bottom bend. Until a definite time has elapsed the current is inappreciable, and then it begins to increase fairly rapidly. As the time increases, e^{-ut}, originally unity, falls. It can fall to some definite value θ_1 before $f(ut)$ has appreciably departed from zero. This value of θ_1 was given by Kelvin as 0·75. Fleeming Jenkin gave as the limiting value 0·79 or $10^{-0\cdot1}$.

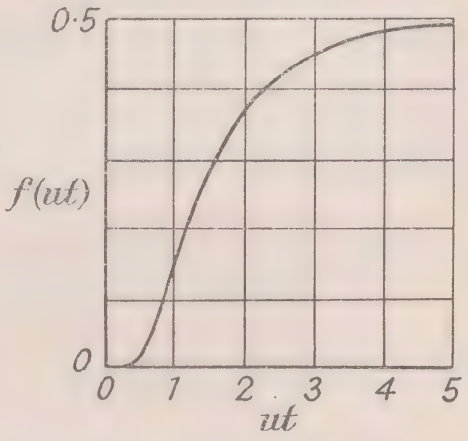


FIG. 91.

Writing $\theta_1 = e^{-ut_1}$

gives $ut_1 = \log \dfrac{1}{\theta_1}$

or $t_1 = \dfrac{1}{u} \log \dfrac{1}{\theta_1} = \dfrac{Cl \,.\, Rl}{\pi^2} \log \dfrac{1}{\theta_1}$

But $Cl = \overline{C}$ the total capacitance of the line and $Rl = \overline{R}$ the total resistance of the line, so that

$$t_1 = \frac{\overline{C} \,.\, \overline{R}}{\pi^2} \log \frac{1}{\theta_1}$$

Taking Fleeming Jenkin's value for θ_1 gives

$$t_1 = \overline{C}\overline{R} \times 0{\cdot}0233$$

During this interval of time no detectable current is received. After a time $t_2 = \overline{C}\overline{R} \times 0{\cdot}14$ the current reaches about one half its final value. After a time $t_3 = \overline{C}\overline{R}$ the current reaches a value inappreciably different from the final value.

CHAPTER XX

FOURIER'S SERIES AND HARMONIC ANALYSIS

It is often required (see for instance the telegraph equations in the previous chapter) to express a function as a series of sines and cosines of an angle and its integral multiples. This can only be done for a finite, univalued function. These conditions existing it is possible over a range of values of x to express the function $y = f(x)$ as a series of the form

$$y = f(x) = a_0 + a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \ldots \quad (733)$$

An alternative form is

$$y = f(x) = c_0 + c_1 \sin (x + \alpha_1) + c_2 \sin (2x + \alpha_2) \ldots \quad (734)$$

If the value of the function is repeated at regular intervals such that

$$f(x) = f(x + 2l) = f(x + 4l) \text{ etc.} \quad (735)$$

it is possible to express the function as the sum of a series of sinoidal waves of fundamental wave-length $2l$.

The problem in general is that of determining the coefficients a_0, a_1 . . . b_1, b_2, etc., or c_0, c_1 etc. Series such as 733 or 734 are known as Fourier Series.

In determining the coefficients use is made of certain integrals. Rough graphs show that if n is an integer the following four integrals are all zero

$$\int_0^{2\pi} \sin n\theta \, d\theta, \quad \int_0^{2\pi} \cos n\theta \, d\theta, \quad \int_{-\pi}^{+\pi} \sin n\theta \, d\theta, \quad \int_{-\pi}^{+\pi} \cos n\theta \, d\theta.$$

Again, writing

$$\cos m\theta \cos n\theta = \tfrac{1}{2}[\cos (m + n)\theta + \cos (m - n)\theta]$$

and

$$\sin m\theta \sin n\theta = \tfrac{1}{2}[\cos (m - n)\theta - \cos (m + n)\theta]$$

it is seen that if m and n are unequal integers then the following four integrals are all zero

$$\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta, \quad \int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta,$$

$$\int_{-\pi}^{+\pi} \cos m\theta \cos n\theta \, d\theta, \quad \int_{-\pi}^{+\pi} \sin m\theta \sin n\theta \, d\theta$$

Again, writing

$$\sin m\theta \cos n\theta = \tfrac{1}{2}[\sin (m + n)\theta + \sin (m - n)\theta]$$

it is seen that so long as m and n are integers, whether equal or not,

$$\int_0^{2\pi} \sin m\theta \cos n\theta \, d\theta = \int_{-\pi}^{+\pi} \sin m\theta \cos n\theta \, d\theta = 0$$

The only integrals of this type which have a finite value are the following four

$$\int_0^{2\pi} \sin^2 n\theta \, d\theta = \int_0^{2\pi} \cos^2 n\theta \, d\theta$$

$$= \int_{-\pi}^{+\pi} \sin^2 n\theta \, d\theta = \int_{-\pi}^{+\pi} \cos^2 n\theta \, d\theta = \pi \quad . \quad (736)$$

It therefore follows that if y is defined by equation 733

$$\int_0^{2\pi} y \, dx = \int_0^{2\pi} a_0 \, dx = a_0 \, . \, 2\pi \text{ giving } a_0 = \frac{1}{2\pi}\int_0^{2\pi} y \, dx \quad . \quad . \quad (737)$$

Or again, if one multiplies by $\sin nx$ then

$$\int_0^{2\pi} y \sin nx \, dx = \int_0^{2\pi} a_n \sin^2 nx \, dx = a_n\pi$$

whence

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} y \sin nx \, dx \quad . \quad . \quad . \quad . \quad (738)$$

and similarly $b_n = \dfrac{2}{2\pi}\displaystyle\int_0^{2\pi} y \cos nx \, dx.$

It must be observed that the series so obtained in general only represents the function over the range of integration. Of course if the function is repeated every 2π, then the series represents the function completely. This is generally the case for alternating current waves.

Examples.

1. Obtain a Fourier series for $y = x$ from 0 to 2π. Assume $x = a_0 + a_1 \sin x + a_2 \sin 2x \ . \ . \ .$

$$+ b_1 \cos x + b_2 \cos 2x \ . \ . \ .$$

Then $a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \pi$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} x \sin nx\,dx = \frac{2}{2\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= -\frac{2}{n}$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} x \cos nx\,dx = \frac{2}{2\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= 0$$

and so $y = \pi - 2\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x \ldots$. (739)

2. Obtain a Fourier series for a function defined by $y = 1 - \frac{x}{\pi}$ between 0 and 2π. The function has a value $+1$ at $x = 0$, -1 at $x = 2\pi$ and zero at $x = \pi$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}\left(1 - \frac{x}{\pi}\right)dx = \frac{1}{2\pi}\left[x - \frac{x^2}{2\pi}\right]_0^{2\pi} = 0$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi}\left(1 - \frac{x}{\pi}\right)\sin nx\,dx$$

$$= \frac{2}{2\pi}\left[-\frac{\cos nx}{n} + \frac{x\cos nx}{\pi n} - \frac{\sin nx}{\pi n^2}\right]_0^{2\pi} = \frac{2}{\pi n}$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi}\left(1 - \frac{x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{2\pi}\left[\frac{\sin nx}{n} - \frac{x\sin nx}{\pi n} - \frac{\cos nx}{\pi n^2}\right]_0^{2\pi} = 0$$

Hence the series is

$$y = \frac{2}{\pi}[\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots] \quad . \quad (740)$$

If a series is required between $x = 0$ and $x = 2l$ then putting $x = \frac{l\theta}{\pi}$, $\theta = \frac{\pi x}{l}$ and as x varies from 0 to $2l$, θ varies from 0 to 2π. This substitution is often very useful.

Fourier's Half Range Series.

A continuous, finite, univalued function of x can be expressed in the alternative form

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x \ . \ . \ . \ . \quad (741)$$

or

$$f(x) = b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x \ . \ . \ . \ . \ . \quad (742)$$

so long as $0 < x < \pi$. These are called respectively Fourier's half-range sine series and half-range cosine series. Two integrals of importance in evaluating the coefficients of these series are

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta \text{ and } \int_0^\pi \cos m\theta \cos n\theta \, d\theta$$

$$\sin m\theta \sin n\theta = \tfrac{1}{2}[\cos (m-n)\theta - \cos (m+n)\theta]$$

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = \tfrac{1}{2}\left[\frac{\sin (m-n)\theta}{m-n} - \frac{\sin (m+n)\theta}{m+n}\right]_0^\pi = 0$$

unless $m = n$.

Similarly it can be shown that

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = 0$$

Hence, as for the full range series

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx \ . \ . \ . \ . \quad (743)$$

and

$$b_0 = \frac{1}{\pi}\int_0^\pi f(x)dx \ . \ . \ . \ . \ . \ . \quad (744)$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx \, dx \ . \ . \ . \ . \quad (745)$$

Example.

Corresponding to $x = 0$ to $x = l$ find a half-range sine series for a function expressed by

$$y = \frac{2}{l}x \text{ for } 0 < x < \frac{l}{2}$$

$$y = \frac{2}{l}(l - x) \text{ for } \frac{l}{2} < x < l \text{ (Fig. 92)}$$

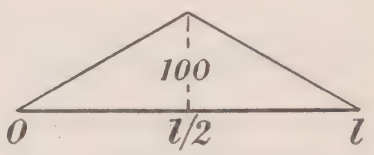


Fig. 92.

Putting $x = \frac{l\theta}{\pi}$ this becomes

$$y = \frac{2\theta}{\pi} \text{ for } 0 < \theta < \frac{\pi}{2}; \quad y = 2\left(1 - \frac{\theta}{\pi}\right) \text{ for } \frac{\pi}{2} < \theta < \pi$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2\theta}{\pi} \sin n\theta \, d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{2}{\pi}(\pi - \theta) \sin n\theta \, d\theta\right]$$

$$= \frac{4}{\pi^2}\left[\int_0^{\frac{\pi}{2}} \theta \sin n\theta \, d\theta + \int_{\frac{\pi}{2}}^{\pi} (\pi - \theta) \sin n\theta \, d\theta\right]$$

If in the second integral one writes $\alpha = \pi - \theta$ then $d\theta = -d\alpha$ and $\sin n\theta = \sin(n\pi - n\alpha)$ and so

$$\int_{\frac{\pi}{2}}^{\pi} (\pi - \theta) \sin n\theta \, d\theta = -\int_{\frac{\pi}{2}}^{0} \alpha \sin(n\pi - n\alpha) d\alpha = \int_0^{\frac{\pi}{2}} \alpha \sin(n\pi - n\alpha) d\alpha$$

If n is even then $\int_0^{\frac{\pi}{2}} \alpha \sin(n\pi - n\alpha) d\alpha = -\int_0^{\frac{\pi}{2}} \alpha \sin n\alpha \, d\alpha$

and so $a_n = 0$ when n is even.

If n is odd $\int_0^{\frac{\pi}{2}} \alpha \sin(n\pi - n\alpha) d\alpha = \int_0^{\frac{\pi}{2}} \alpha \sin n\alpha \, d\alpha$ and

$$a_n = \frac{8}{\pi^2}\int_0^{\frac{\pi}{2}} \theta \sin n\theta \, d\theta = \frac{8}{\pi^2}\left[-\frac{\theta \cos n\theta}{n} + \frac{\sin n\theta}{n^2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2} \text{ giving } a_1 = \frac{8}{\pi^2}, \; a_3 = -\frac{8}{\pi^2} \cdot \frac{1}{3^2} \cdot \ldots$$

Hence $y = \frac{8}{\pi^2}\left[\sin\theta - \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta - \ldots\right]$.

or

$$y = \frac{8}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} \ldots\right] \quad \ldots \quad (746)$$

This is the expansion used in the previous chapter (equation 719).

Harmonic Curves.

A function which is repeated at regular intervals is said to be harmonic. If $2l$ is the shortest distance in which the function is repeated then a convenient substitution is that

already suggested, $\theta = \frac{\pi x}{l}$. The function may then be expressed as

$$y = a_0 + a_1 \sin\theta + a_2 \sin 2\theta \ . \ . \ . + b_1 \cos\theta + b_2 \cos 2\theta \ . \ . \ .$$

or

$$y = a_0 + a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} \ . \ . \ . + b_1 \cos\frac{\pi x}{l} + b_2 \cos\frac{2\pi x}{l} \ . \ . \ .$$

When the graph of the function possesses any degree of symmetry a proper choice of the origin enables the function to be expressed in its simplest form. Thus the simple sinoidal function of Fig. 93 is repeated continually every interval of $2l$.

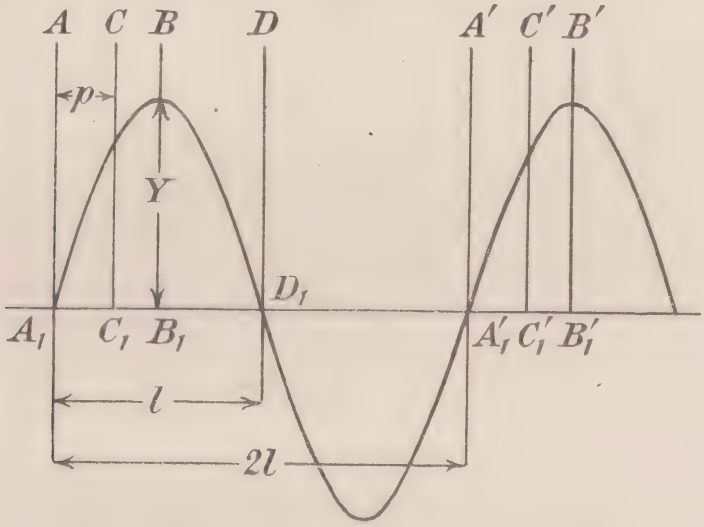


FIG. 93.

Choosing the origin as A_1 the function is represented by $y = Y \sin\frac{\pi x}{l}$. Choosing the origin as B_1 the function is represented by $y = Y \cos\frac{\pi x}{l}$. These two forms are equally simple and one would naturally adopt one of them. If C_1 is taken as the origin then the function is represented by

$$y = Y \sin\frac{\pi(x+p)}{l} = Y_1 \sin\frac{\pi x}{l} + Y_2 \cos\frac{\pi x}{l}$$

where $Y_1 = Y \cos\frac{\pi p}{l}$ and $Y_2 = Y \sin\frac{\pi p}{l}$, a much more complicated expression. The point should be noted that between BB_1 and $B'B_1'$ the function is symmetrical but that between AA_1 and $A'A_1'$ it is not. However, the half wave-lengths

AA_1 to DD_1 and DD_1 to $A'A_1'$ are separately symmetrical. Symmetry of this kind may be conveniently investigated by expressing it in a half-range series. The consideration of this form of representation will be deferred until other wave forms have been considered.

In many practical cases the constant a_0 does not exist. In every case it may be suppressed by choice of the horizontal axis so that the area above is equal to the area below it.

General wave forms may be highly complicated and yet possess a certain degree of symmetry. Analysis of these wave forms may be much simplified once the significance of the

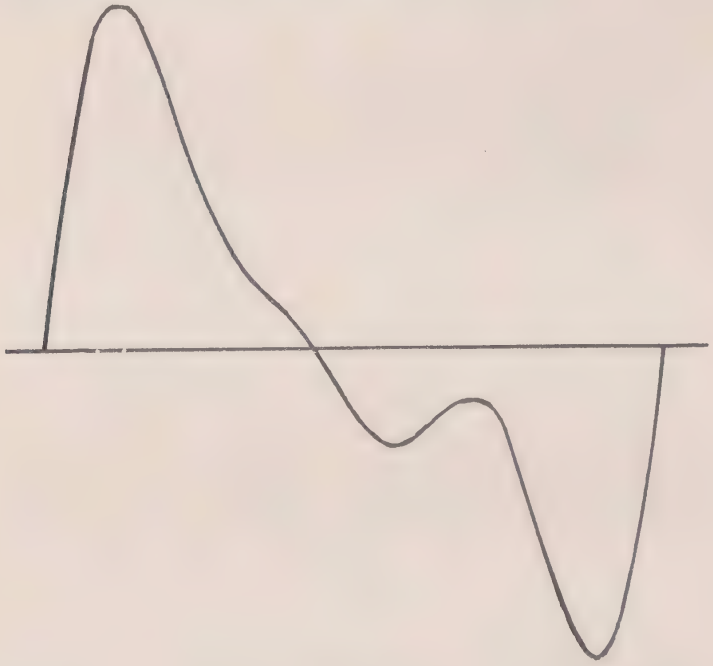


Fig. 94.

particular type of symmetry is recognised. Fortunately the engineer has rarely to deal with wave forms in which no symmetry is recognisable. If there is none, as in the wave form of Fig. 94, no special guide to procedure can be given.

One important type of symmetry is represented in Fig. 95. The figure is symmetrical between AA_1 and $A'A_1'$ and is also symmetrical between BB_1 and $B'B_1'$. The positive and negative parts of the wave, though very different in form, are equal in area. The curve is symmetrical with respect to AA_1 or BB_1. If A_1 be selected as the origin then any point at a distance θ from A_1 is represented by

$$y = a_1 \sin\theta + a_2 \sin 2\theta + \ldots + b_1 \cos\theta + b_2 \cos 2\theta + \ldots$$

The ordinate at a point equally to the left of A_1 is

$$y' = -a_1 \sin\theta - a_2 \sin 2\theta - \ldots + b_1 \cos\theta + b_2 \cos 2\theta + \ldots$$

But since the curve is symmetrical about AA_1, $y = y'$ and

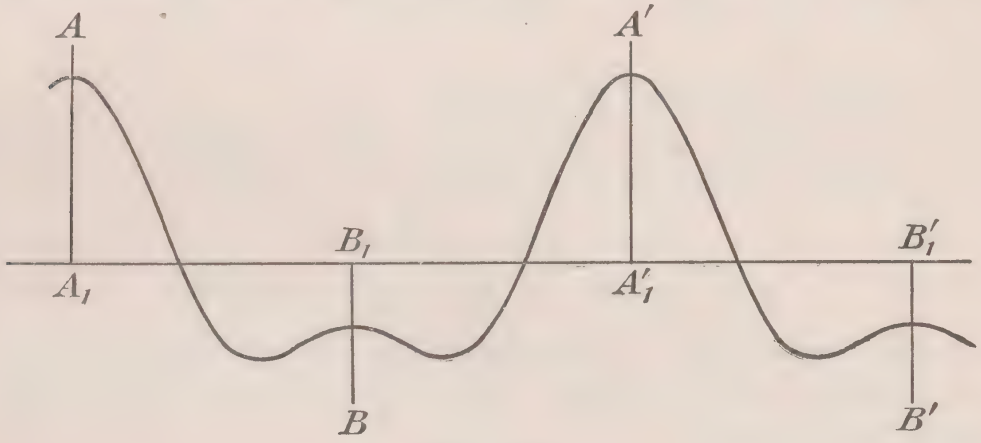


Fig. 95.

$a_1 = a_2 = a_3$ etc. $= 0$, and the function is completely represented by the cosine series

$$y = b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta \ldots \text{etc.} \quad . \quad . \quad . \quad (747)$$

This is the most convenient form in which to express a curve

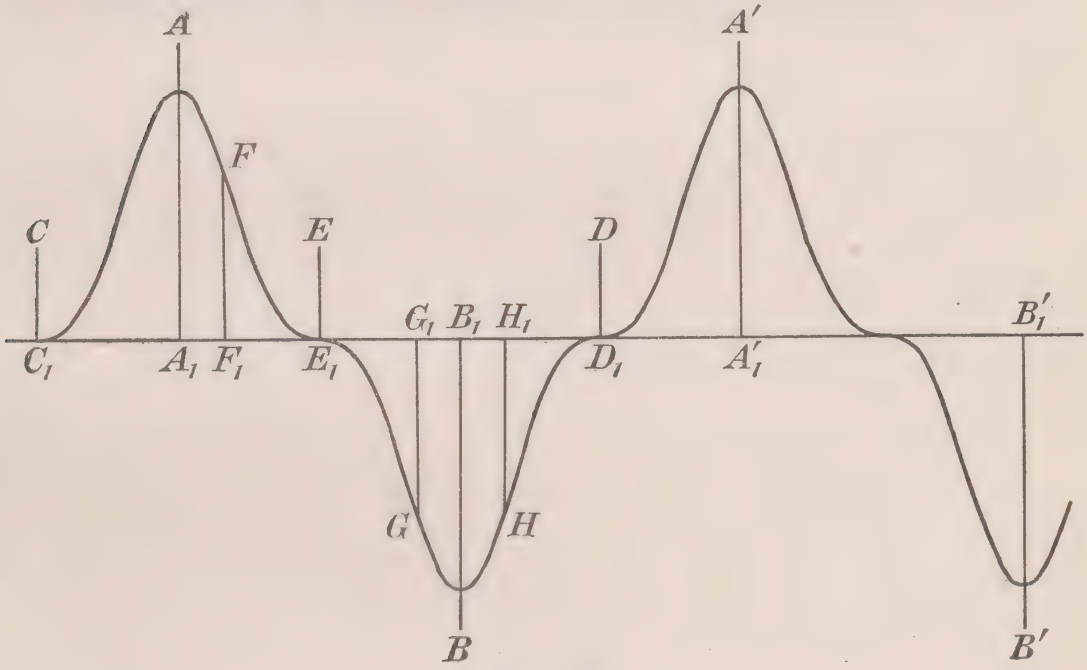


Fig. 96.

such as Fig. 95. A greater degree of symmetry is exhibited by Fig. 96. Here again the figure is symmetrical between AA_1 and $A'A_1'$ and between BB_1 and $B'B_1'$, and so the equation will not be more complicated than 747 if A_1 is chosen

as the origin. A further simplification is however possible, for in this case not only are the positive and negative portions of the wave equal in area, but they are identical and symmetrical in the sense that $BB_1 = -AA_1$ and $FF_1 = -HH_1 = -GG_1$ or

$$\begin{aligned} y &= b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta \ldots \\ &= -[b_1 \cos(\theta+\pi) + b_2 \cos 2(\theta+\pi) + b_3 \cos 3(\theta+\pi) \ldots] \\ &= b_1 \cos\theta - b_2 \cos 2\theta + b_3 \cos 3\theta - b_4 \cos 4\theta + \ldots \end{aligned}$$

whence $$b_2 = b_4 = b_6 \text{ etc.} = 0$$

and $$y = b_1 \cos\theta + b_3 \cos 3\theta + b_5 \cos 5\theta + \ldots \quad (748)$$

Only odd terms occur. There is in this case another convenient representation of the curve. If C_1 be selected as the origin instead of A_1 then

$$y = b_1 \cos\left(\theta - \frac{\pi}{2}\right) + b_3 \cos 3\left(\theta - \frac{\pi}{2}\right) + b_5 \cos 5\left(\theta - \frac{\pi}{2}\right) + \ldots$$

or

$$y = b_1 \sin\theta - b_3 \sin 3\theta + b_5 \sin 5\theta - b_7 \sin 7\theta + \text{etc.} \quad (749)$$

a sine series with only odd terms occurring.

Two important types of symmetry are shown in Figs. 97

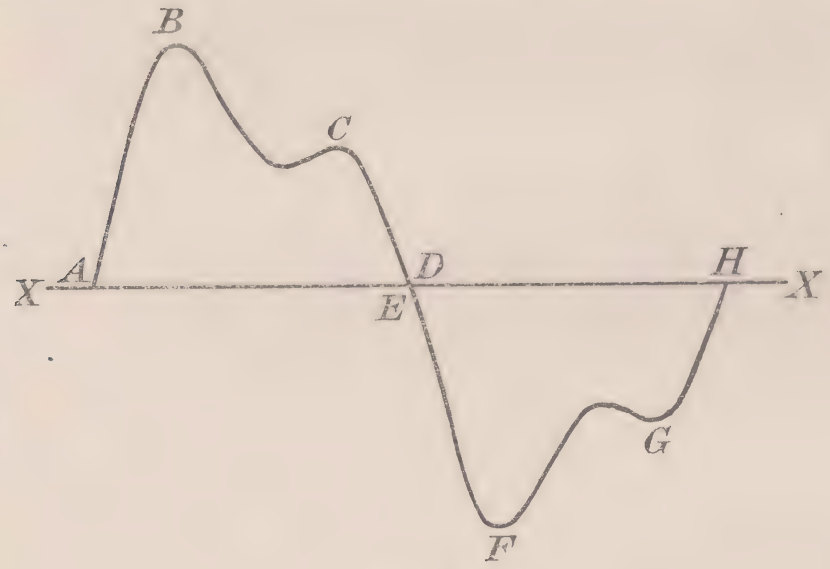


FIG. 97.

and 98. The symmetry of Fig. 97 is fundamentally that of the alternating current supplied under certain circumstances by a multipolar generator. If the negative part of the curve is imagined moved to the left by an amount π so that E coincides with A then the two portions are mirror images of one another

in the axis XX. If the ordinates are represented by

$$y = a_1 \sin\theta + a_2 \sin 2\theta + a_3 \sin 3\theta + \ldots + b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta + \ldots$$

then $$y' = f(\pi + \theta) = -f(\theta)$$

or $$a_1 \sin\theta + a_2 \sin 2\theta + a_3 \sin 3\theta + \ldots + b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta + \ldots = -[-a_1 \sin\theta + a_2 \sin 2\theta - a_3 \sin 3\theta + \ldots] - [-b_1 \cos\theta + b_2 \cos 2\theta - b_3 \cos 3\theta + \ldots]$$

giving $a_2 = a_4 = a_6 \ldots = 0$ and $b_2 = b_4 = b_6 \ldots = 0$

Only the odd harmonics occur and the equation is

$$y = a_1 \sin\theta + a_3 \sin 3\theta + a_5 \sin 5\theta \ldots + b_1 \cos\theta + b_3 \cos 3\theta + b_5 \cos 5\theta \ldots \quad (750)$$

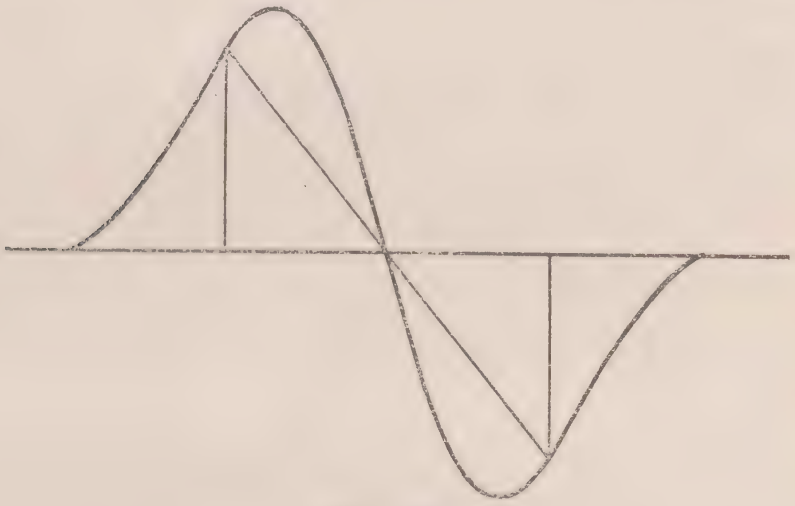


FIG. 98.

The symmetry of Fig. 98 is quite different and is less important. Any positive part of the curve is the point image of the adjacent negative portion through the connecting node.

$$y = f(\theta) = -f(2\pi - \theta)$$

So that $$a_1 \sin\theta + a_2 \sin 2\theta + a_3 \sin 3\theta \ldots + b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta \ldots = -[-a_1 \sin\theta - a_2 \sin 2\theta - a_3 \sin 3\theta \ldots] - [b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta \ldots]$$

whence $b_1 = b_2 = b_3$ etc. $= 0$. Only sine terms occur if the origin is a node and the equation is

$$y = a_1 \sin\theta + a_2 \sin 2\theta + a_3 \sin 3\theta + \ldots \quad (751)$$

It will be noted that a completely symmetrical curve such as Fig. 96 possesses both the symmetry of Fig. 97 and of Fig. 98.

Therefore if the origin is a node the equation contains only odd sine terms. This being the case, if it is desired to obtain a Fourier series for a completely symmetrical curve, such as Fig. 96, the labour may be reduced by determining the half-range sine series fitting the positive half of the wave. This half-range sine series is the complete equation and is not limited in application only to the half-range for which it is determined. Finding the coefficients is simplified by the fact that n can have only odd integral values.

The Trapezoidal Wave.

The trapezoidal wave is of importance in determining the flux distribution in the air gap of an alternator. The stator winding is usually so distributed that the magnetomotive force is represented approximately by a graph of the form shown in Fig. 99. It is necessary to find a half-range sine series for the

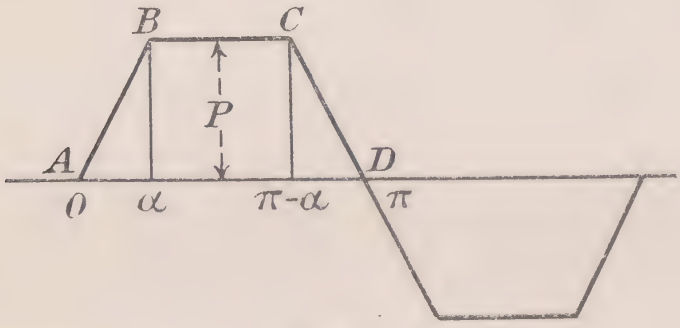


FIG. 99.

portion $ABCD$. This will then be the equation for the whole curve. The equation is

$$y = a_1 \sin\theta + a_3 \sin 3\theta + a_5 \sin 5\theta + \ldots \quad (752)$$

The curve is described as

(i) $0 < \theta < \alpha$, $y = P\theta/\alpha$; (ii) $\alpha < \theta < \pi - \alpha$, $y = P$;
(iii) $\pi - \alpha < \theta < \pi$, $y = P(\pi - \theta)/\alpha$

$$a_n = \frac{2}{\pi}\int_0^{\pi} y \sin n\theta \, d\theta$$

or

$$a_n = \frac{2}{\pi}\left[\int_0^{\alpha}\frac{P\theta}{\alpha}\sin n\theta \, d\theta + \int_{\alpha}^{\pi-\alpha} P \sin n\theta \, d\theta + \int_{\pi-\alpha}^{\pi}\frac{P(\pi-\theta)}{\alpha}\sin n\theta \, d\theta\right]$$

If $\beta = \pi - \theta$, $d\theta = -d\beta$, $\sin n\theta = \sin(n\pi - n\beta)$; since

n is odd, $\sin n\theta = \sin n\beta$; when $\theta = \pi, \beta = 0$; when $\theta = \pi - \alpha$, $\beta = \alpha$. Hence

$$\int_{\pi-\alpha}^{\pi} \frac{P(\pi - \theta)}{\alpha} \sin n\theta \, d\theta = \int_0^{\alpha} \frac{P\beta}{\alpha} \sin n\beta \, d\beta = \int_0^{\alpha} \frac{P\theta}{\alpha} \sin n\theta \, d\theta$$

$$\text{Hence } a_n = \frac{2}{\pi}\left[2\int_0^{\alpha} \frac{P\theta}{\alpha} \sin n\theta \, d\theta + \int_{\alpha}^{\pi-\alpha} P \sin n\theta \, d\theta \right]$$

$$\text{or} \quad a_n = \frac{2}{\pi}\left[2\frac{P}{\alpha}\left\{ -\frac{0 \cos n0}{n} + \frac{\sin n\theta}{n^2} \right\}_0^{\alpha} - \frac{P}{n}(\cos n\theta)_{\alpha}^{\pi-\alpha} \right]$$

$$\text{or} \quad a_n = \frac{4P}{\pi} \frac{\sin n\alpha}{\alpha n^2}$$

Hence

$$y = \frac{4P}{\pi\alpha}\left[\sin\alpha \sin\theta + \frac{\sin 3\alpha \sin 3\theta}{3^2} + \frac{\sin 5\alpha \sin 5\theta}{5^2} + \ldots \right] \qquad (753)$$

Rotating Wave of Flux in Air Gap of a 3-Phase Machine.

In a 3-phase machine three windings, each of which would give a flux distribution such as Fig. 99, if fed with direct current, are displaced in space by 120 electrical degrees. They are fed with equal 3-phase currents. The angle α in equation 753 is $\frac{\pi}{6}$. Writing $P = Q \cos \omega t$ for the first winding, $Q \cos\left(\omega t - \frac{2\pi}{3}\right)$ for the second winding, and $Q \cos\left(\omega t - \frac{4\pi}{3}\right)$ for the third winding, one obtains (on inserting $\frac{\pi}{6}$ for α)

$$y_1 = \frac{24Q}{\pi^2} \cos \omega t \sum_{n=1,\,3\ldots}^{\infty} \frac{1}{n^2}\left[\sin \frac{n\pi}{6} \sin n\theta \right]$$

$$y_2 = \frac{24Q}{\pi^2} \cos\left(\omega t - \frac{2\pi}{3}\right) \sum_{n=1,\,3\ldots}^{\infty} \frac{1}{n^2}\left[\sin \frac{n\pi}{6} \sin n\left(\theta - \frac{2\pi}{3}\right) \right]$$

$$y_3 = \frac{24Q}{\pi^2} \cos\left(\omega t - \frac{4\pi}{3}\right) \sum_{n=1,\,3\ldots}^{\infty} \frac{1}{n^2}\left[\sin \frac{n\pi}{6} \sin n\left(\theta - \frac{4\pi}{3}\right) \right]$$

If n is a multiple of 3

$$\sin n\theta = \sin n\left(\theta - \frac{2\pi}{3}\right) = \sin n\left(\theta - \frac{4\pi}{3}\right)$$

and since $\cos \omega t + \cos\left(\omega t - \frac{2\pi}{3}\right) + \cos\left(\omega t - \frac{4\pi}{3}\right) = 0.$

$y_1 + y_2 + y_3 = 0$ when n is a multiple of 3.

Since only odd terms occur the only values of n left are of the form $6p \pm 1$. When $n = 6p \pm 1$

$$\sin n\left(\theta - \frac{2\pi}{3}\right) = \sin n\theta \mp \frac{2\pi}{3}$$

$$\sin n\left(\theta - \frac{4\pi}{3}\right) = \sin n\theta \mp \frac{4\pi}{3}$$

Taking first $n = 6p + 1$

$$\cos \omega t \sin n\theta = \frac{1}{2}[\sin (n\theta + \omega t) + \sin (n\theta - \omega t)]$$

$$\cos\left(\omega t - \frac{2\pi}{3}\right)\sin\left(n\theta - \frac{2\pi}{3}\right) = \frac{1}{2}\left[\sin\left(n\theta + \omega t - \frac{4\pi}{3}\right) + \sin(n\theta - \omega t)\right]$$

$$\cos\left(\omega t - \frac{4\pi}{3}\right)\sin\left(n\theta - \frac{4\pi}{3}\right) = \frac{1}{2}\left[\sin\left(n\theta + \omega t - \frac{8\pi}{3}\right) + \sin(n\theta - \omega t)\right]$$

and the sum of these three is $\frac{3}{2}\sin (n\theta - \omega t)$.

Taking now $n = 6p - 1$

$$\cos \omega t \sin n\theta = \frac{1}{2}[\sin (n\theta + \omega t) + \sin (n\theta - \omega t)]$$

$$\cos\left(\omega t - \frac{2\pi}{3}\right)\sin\left(n\theta + \frac{2\pi}{3}\right) = \frac{1}{2}\left[\sin(n\theta + \omega t) + \sin\left(n\theta - \omega t + \frac{4\pi}{3}\right)\right]$$

$$\cos\left(\omega t - \frac{4\pi}{3}\right)\sin\left(n\theta + \frac{4\pi}{3}\right) = \frac{1}{2}\left[\sin(n\theta + \omega t) + \sin\left(n\theta - \omega t + \frac{8\pi}{3}\right)\right]$$

and the sum of these three is $\frac{3}{2}\sin (n\theta + \omega t)$.

When $n = 1$, $\sin \frac{n\pi}{6} = \frac{1}{2}$; when $n = 5$, $\sin \frac{n\pi}{6} = \frac{1}{2}$;

when $n = 7$, $\sin \frac{n\pi}{6} = -\frac{1}{2}$; when $n = 11$, $\sin \frac{n\pi}{6} = -\frac{1}{2}$;

when $n = 13$, $\sin \frac{n\pi}{6} = \frac{1}{2}$ and so on, and so

$$y = y_1 + y_2 + y_3 = \frac{18Q}{\pi^2}\left[\sin (\theta - \omega t) + \frac{\sin (5\theta + \omega t)}{5^2} - \frac{\sin (7\theta - \omega t)}{7^2} - \frac{\sin (11\theta + \omega t)}{11^2} + \ldots\right] \quad (754)$$

The fundamental rotating field has an electrical angular velocity

of ω, the fifth harmonic field an angular velocity $\frac{\omega}{5}$ in the opposite direction, the seventh harmonic field an angular velocity $\frac{\omega}{7}$ in the same direction as the fundamental, and so on.

It must not be imagined that a conductor immersed in this field would have an e.m.f. generated in it containing harmonics. If equation 754 be differentiated with respect to t it is seen that the rate of change of y at any place is purely sinoidal. From a physical point of view it is obvious that this must be so, for the conductor is immersed in a field made up of three purely sinoidal components. Or again, from equation 754, as the field revolves each component generates an e.m.f. The n'th harmonic pole pitch is only one n'th of the pitch of the fundamental field, but it is moving at one n'th of the speed and therefore generates an e.m.f. of the same frequency as that generated by the fundamental field.

If $P = R.M.S$ ampere turns per pole per phase, $Q = \sqrt{2}P$. The maximum value of the magnetomotive force of the fundamental wave in the air gap, measured in ampere turns is (from equation 754) $\frac{18Q}{\pi^2} = \frac{18\sqrt{2}P}{\pi^2}$. The average magnetomotive force of the fundamental over the whole pole pitch is $\frac{2}{\pi} \cdot \frac{18\sqrt{2}}{\pi^2}P = \frac{36\sqrt{2}}{\pi^3}P = 1{\cdot}641P$. This is the value usually assumed (from graphical approximations) in estimating the effects of armature reaction.

Rectified Sine Wave.

An important wave form in view of the modern use of rectifiers is the rectified sine wave shown in Fig. 100. This has a direct component b_0 which is definitely required. Superposed on this is an alternating voltage of fundamental frequency twice that of the supply. It is often effectively reduced to a low value by means of smoothing circuits. It is seen that the wave form has the symmetry characteristics of Fig. 95. Taking

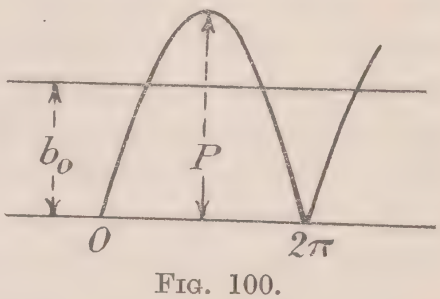


Fig. 100.

the origin at O the Fourier series may be expressed as

$$y = b_0 + b_1 \cos\theta + b_2 \cos 2\theta + b_3 \cos 3\theta \ . \ . \ . \text{ etc.}$$

Between 0 and 2π, y may be expressed as

$$y = P \sin\frac{\theta}{2}$$

then

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} P \sin\frac{\theta}{2}\, d\theta = \frac{2}{\pi}P$$

$$b_n = \frac{2P}{2\pi}\int_0^{2\pi} \sin\frac{\theta}{2} \cos n\theta \, d\theta$$

Writing $\theta = 2\alpha$, $d\theta = 2d\alpha$ and

$$\int_0^{2\pi} \sin\frac{\theta}{2} \cos n\theta \, d\theta = \int_0^{\pi} 2 \sin\alpha \cos 2n\alpha d\alpha$$

$$= \int_0^{\pi} [\sin(2n+1)\alpha - \sin(2n-1)\alpha] d\alpha$$

$$= {}_0^{\pi}\left[\frac{\cos(2n-1)\alpha}{2n-1} - \frac{\cos(2n+1)\alpha}{2n+1}\right]$$

$$= \frac{2}{2n+1} - \frac{2}{2n-1} = -\frac{4}{4n^2-1}$$

whence

$$y = \frac{4P}{\pi}\left[\frac{1}{2} - \frac{\cos\theta}{3} - \frac{\cos 2\theta}{15} - \frac{\cos 3\theta}{35} \ . \ . \ . - \frac{\cos n\theta}{4n^2-1} \ . \ . \ .\right] \quad . \quad (755)$$

It is seen that the second harmonic is 20 per cent. of the fundamental.

Harmonic Analysis.

In some cases it is necessary to determine from an oscillogram, or from a graph obtained by use of a Joubert contact, the equation to the wave form of an alternating voltage or current. Many methods are available of varying accuracy. From the curve a number of representative points are picked off, conveniently uniformly spaced along the time, or angle, axis. The more ordinates that are taken, the more accurate is the description of the curve. What perhaps is most important is accuracy of draughtsmanship and measurement.

The most direct method of using the measurements so obtained is based upon the general theory of Fourier analysis given at the beginning of this chapter.

Assuming that the curve is expressible as

$$y = a_0 + a_1 \sin\theta + a_2 \sin 2\theta \ldots + b_1 \cos\theta + b_2 \cos 2\theta \ldots$$

then $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} y\, d\theta$$

that is to say a_0 is the average height of the figure or the mean of the ordinates selected.

Again $$a_n = \frac{2}{2\pi}\int_0^{2\pi} y \sin n\theta\, d\theta$$

To determine a_n therefore each ordinate is multiplied by $\sin n\theta$ (θ having its particular value for each ordinate). The average of these new products so obtained is $\frac{1}{2\pi}\int_0^{2\pi} y \sin n\theta\, d\theta$ or $\frac{a_n}{2}$. b_n is determined in a similar manner.

The method may be illustrated by a simple example. It must be understood that in a practical case more ordinates should be taken if an accurate description of the curve is required.

Example. Analyse the wave form described by the following measurements (this is an inversion of the curve plotted in Fig. 94) :

$\theta°$	0°	30°	60°	90°	120°	150°
y	134	444	326	106	94	144
$\theta°$	180°	210°	240°	270°	300°	330°
y	66	− 44	− 126	− 306	− 494	− 344

The sum of these ordinates is zero and so $a_0 = 0$. To determine the successive coefficients a table is set out as below

$\theta°$	y	$\sin\theta$	$y\sin\theta$	$\cos\theta$	$y\cos\theta$	$\sin 2\theta$	$y \sin 2\theta$	$\cos 2\theta$	$y \cos 2\theta$	$\sin 3\theta$	$y \sin 3\theta$	$\cos 3\theta$	$y \cos 3\theta$
0	134	0	0	1	134	0	0	1	134	0	0	1	134
30	444	0·5	222	0·866	384	0·866	384	0·5	222	1	444	0	0
60	326	0·866	282	0·5	163	0·866	282	−0·5	−163	0	0	−1	−326
90	106	1	106	0	0	0	0	−1	−106	−1	−106	0	0
120	94	0·866	81	−0·5	− 47	−0·866	− 81	−0·5	− 47	0	0	1	94
150	144	0·5	72	−0·866	−125	−0·866	−125	0·5	72	1	144	0	0
180	66	0	0	−1	− 66	0	0	1	66	0	0	−1	− 66
210	− 44	−0·5	22	−0·866	38	0·866	− 38	0·5	− 22	−1	44	0	0
240	−126	−0·866	109	−0·5	63	0·866	−109	−0·5	63	0	0	1	−126
270	−306	−1	306	0	0	0	0	−1	306	1	−306	0	0
300	−494	−0·866	428	0·5	−247	−0·866	428	−0·5	247	0	0	−1	494
330	−344	−0·5	172	0·866	−298	−0·866	298	0·5	−172	−1	344	0	0
Totals			1800		− 1		1039		600		564		204
2× mean			300		− 0·2		173·2		100		94		34

Thus up to the third harmonic and within the limits of accuracy possible the series is

$$y = 300 \sin \theta + 173 \sin 2\theta + 100 \cos 2\theta + 94 \sin 3\theta + 34 \cos 3\theta$$

If it is necessary to determine whether there are residual components left of any magnitude it is necessary to construct a table as the work progresses of the value of each component A running balance can then be kept. The work is completed when this balance is zero within the limits of accuracy.

When the curve is of the form of Fig. 97, containing only odd harmonics, it is only necessary to work from 0 to π. This will be evident on consideration of the following integrals

$$\int_0^{\pi} \cos (2m + 1)\theta \cos (2n + 1)\theta \, d\theta$$

$$= \int_0^{\pi} \frac{1}{2}[\cos 2(m - n)\theta + \cos 2(m + n)\theta] d\theta = 0$$

unless $m = n$. Then its value is $\frac{\pi}{2}$ which is

$$\frac{1}{2}\int_0^{2\pi} \cos (2m + 1)\theta \cos (2m + 1)\theta \, d\theta$$

Similarly it can be shown that

$$\int_0^{\pi} \sin (2m + 1)\theta \sin (2n + 1)\theta \, d\theta$$

is zero unless $m = n$ when its value is $\frac{\pi}{2}$ or

$$\frac{1}{2}\int_0^{2\pi} \sin (2m + 1)\theta \sin (2m + 1)\theta \, d\theta$$

Again

$$\int_0^{\pi} \cos (2m + 1)\theta \sin (2n + 1)\theta \, d\theta$$

is shown to be always zero in the same way. It therefore follows that it is only necessary to integrate from 0 to π instead of from 0 to 2π in determining the coefficients. The relevant products are simply repeated from π to 2π.

As an example another wave form will be considered—that of Fig. 97 which only contains a third harmonic in addition to the fundamental.

The values are given in the columns of the following table.

$\theta°$	y	$\sin\theta$	$y\sin\theta$	$\cos\theta$	$y\cos\theta$	$\sin 3\theta$	$y\sin 3\theta$	$\cos 3\theta$	$y\cos 3\theta$
0	−87	0	0	1	− 87	0	0	1	−87
20	103	0·342	35	0·940	97	0·866	89	0·5	51
40	280	0·643	180	0·766	214	0·866	242	−0·5	−140
60	347	0·866	300	0·500	174	0	0	−1	−347
80	295	0·985	291	0·174	51	−0·866	−255	−0·5	−147
100	208	0·985	205	−0·174	− 36	−0·866	−180	0·5	104
120	173	0·866	150	−0·500	− 86	0	0	1	173
140	193	0·643	124	−0·766	−148	0·866	167	0·5	96
160	190	0·342	65	−0·940	−179	0·866	164	−0·5	− 95
Totals . . .			1350		0		227		−392
2 × mean . .			300		0		50·5		−87·2

Thus $y = 300\sin\theta + 50\sin 3\theta - 87\cos 3\theta$

Other Methods.

There are several other methods of attacking the problem. One method due to Kemp is of interest. Suppose the equation to the wave to be represented by

$$y_\theta = c_1\sin(\theta+\alpha_1) + c_3\sin(3\theta+\alpha_3) + c_5\sin(5\theta+\alpha_5)\ \ldots$$

then the ordinate at the angle $-\theta$ is

$$y_{-\theta} = c_1\sin(-\theta+\alpha_1) + c_3\sin(-3\theta+\alpha_3) + c_5\sin(-5\theta+\alpha_5)\ \ldots$$

then $y_\theta + y_{-\theta} = 2c_1\sin\alpha_1\cos\theta + 2c_3\sin\alpha_3\cos 3\theta + \ \ldots$

and $y_\theta - y_{-\theta} = 2c_1\cos\alpha_1\sin\theta + 2c_3\cos\alpha_3\sin 3\theta + \ \ldots$

By giving different values to θ simultaneous equations are obtained for $c_1\sin\alpha_1$, $c_3\sin\alpha_3$ . . . and $c_1\cos\alpha_1$, $c_3\cos\alpha_3$ . . . The equations are solved for these quantities.

Then $$c_n = \sqrt{(c_n\cos\alpha_n)^2 + (c_n\sin\alpha_n)^2}$$

and $$\tan\alpha_n = \frac{c_n\sin\alpha_n}{c_n\cos\alpha_n}$$

To solve to the seventh harmonic, say, eight quantities are required, namely, c_1, c_3, c_5, c_7 and α_1, α_3, α_5, α_7. To determine these, eight simultaneous equations are required, obtained from eight ordinates. The method will be illustrated by a very simple case, the analysis of a curve—that tabulated above—to the third harmonic. For this 4 ordinates are required. These may be equispaced, but that is not essential. They must, however, be symmetrical about $\frac{\pi}{2}$ for use is made of the

fact that $y_{-\theta} = -y_{\pi-\theta}$. The selected values are: $\theta = 20°$, $y = 103$; $\theta = -20°$, $y = -190$ since when $\theta = 160°$, $y = 190$; $\theta = 60°$, $y = 347$; $\theta = -60°$, $y = -173$. Then

$$y_{20} - y_{-20} = 293 = 2c_1 \cos\alpha_1 \sin 20° + 2c_3 \cos\alpha_3 \sin 60°$$
$$y_{60} - y_{-60} = 520 = 2c_1 \cos\alpha_1 \sin 60° + 2c_3 \cos\alpha_3 \sin 180°$$

or
$$293 = 0{\cdot}684c_1 \cos\alpha_1 + 1{\cdot}732c_3 \cos\alpha_3$$
$$520 = 1{\cdot}732\, c_1 \cos\alpha_1$$

giving $c_1 \cos\alpha_1 = 300{\cdot}2$, $c_3 \cos\alpha_3 = 50{\cdot}5$.

Again

$$y_{20} + y_{-20} = -87 = 2c_1 \sin\alpha_1 \cos 20° + 2c_3 \sin\alpha_3 \cos 60°$$
$$y_{60} + y_{-60} = 174 = 2c_1 \sin\alpha_1 \cos 60° + 2c_3 \sin\alpha_3 \cos 180°$$

or
$$-87 = 1{\cdot}880\, c_1 \sin\alpha_1 + c_3 \sin\alpha_3$$
$$174 = c_1 \sin\alpha_1 - 2c_3 \sin\alpha_3$$

giving $c_1 \sin\alpha_1 = 0$ and $c_3 \sin\alpha_3 = -87$, $\alpha_1 = 0$, $c_1 = 300{\cdot}2$, $c_3 = 100{\cdot}5$, $\tan\alpha_3 = -1{\cdot}72$, $\alpha_3 = -60°$

giving $y = 300 \sin\theta + 100 \sin(3\theta - 60°)$

within the limits of accuracy of the work.

Facility with determinants is of considerable value when solving the simultaneous equations.

The Ninth Harmonic.

If the ninth harmonic is of importance care should be taken in the selection of ordinates. If there is no common factor

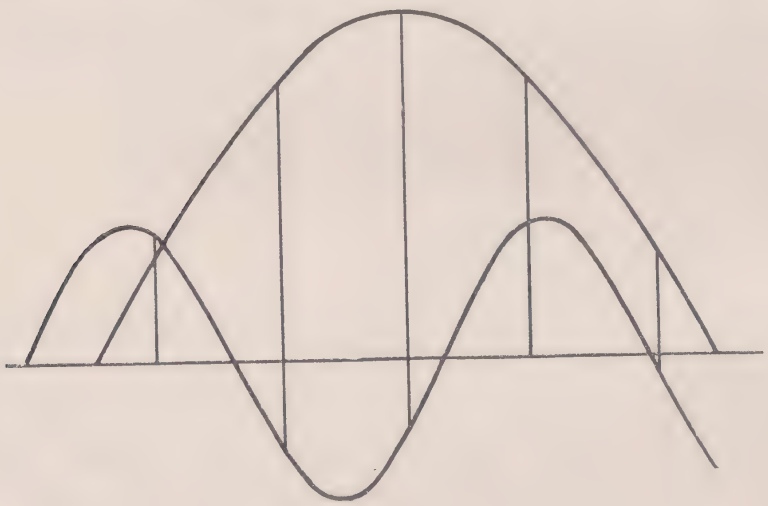


Fig. 101.

between the number of the harmonic and the number of ordinates then in effect the harmonic is served with as many ordinates as the fundamental. This is seen in Fig. 101 where

five ordinates cut the third harmonic at five different points. But if ordinates are placed at every 20° it is conceivable, as shown in Fig. 102, that the ordinates might cut the curve at the nodes of the ninth harmonic. No indication of the presence, even, of the ninth harmonic would then be given by the values of the ordinates. In any case the magnitude and phase of the ninth harmonic cannot be determined by ordinates spaced 20° apart.

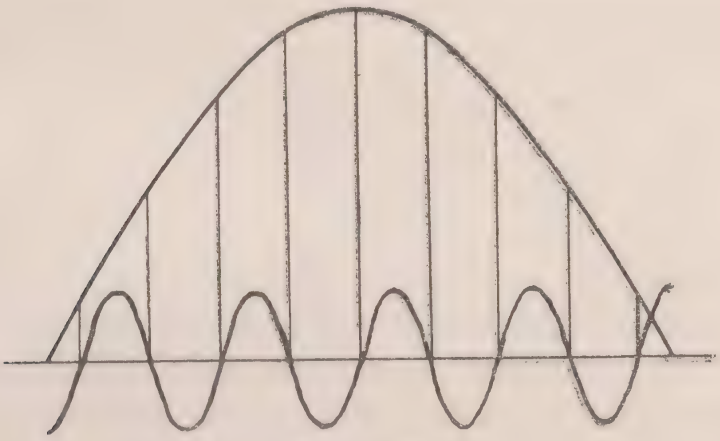


Fig. 102.

The value of a single comprehensive method for determining both the lower and higher harmonics is open to question unless great care is exercised in placing and measuring the ordinates. The author prefers to use a fairly simple scheme for determining the lower harmonics and then to subtract the result so obtained from the original curve. This need only be done for a small portion where the slope is small. The residual is then tackled as a separate problem.

CHAPTER XXI

HEAVISIDE'S OPERATIONAL CALCULUS

A large part of Heaviside's published work on electromagnetic phenomena is devoted to the discussion of disturbances in electrical systems. The underlying principles of electromagnetics had been treated most exhaustively by Maxwell. From a theoretical point of view the problems which practice offered did not require the enunciation of any new principle, but the application of conventional mathematics to such problems presented difficulties of considerable magnitude. The differential equations involved were often of quite simple types, most frequently linear differential equations with constant coefficients. The difficulties arose in their application to discontinuous functions. Heaviside set himself the problem of discovering some new avenue of approach which would lead to the desired solutions. The result was the operational calculus known by his name.

Unfortunately he himself gives no adequate discussion of his methods. Their application was to him more important. Processes were used without any real indication of their validity, and often it is difficult to follow in his papers the way in which he arrived, apparently simply, at his results.

Naturally his methods have been subjected to much critical comment by the more rigid mathematicians. But as they obtain results with greater ease than can be obtained by more orthodox mathematics they have naturally attracted the attention of engineers. It is now possible to approach the operational calculus by methods of unimpeachable rigidity, but the present writer is of opinion that what was probably Heaviside's own approach is easier.

The Unit Function.

It has been seen that the solution of a linear differential equation with constant coefficients usually resolves itself into

the solution of a number of linear differential equations of the first order of the form

$$\frac{dy}{dt} - ay = f(t) \quad . \quad . \quad . \quad . \quad . \quad (756)$$

The solution to this is

$$y = Ae^{at} + e^{at}\int e^{-at}f(t)dt \quad . \quad . \quad . \quad . \quad (757)$$

Using operational methods the two components were determined separately as (a) the complementary function $A\varepsilon^{at} = \frac{1}{p-a} \cdot 0$ and (b) the particular integral

$$e^{at}\int e^{-at}f(t)dt = \frac{1}{p-a} \cdot f(t).$$

Two methods of determining $\frac{1}{p-a} \cdot f(t)$ were found by expanding $\frac{1}{p-a}$ either as (see equations 317 and 318, p. 167) :

$$\frac{1}{p-a} = \left(\frac{1}{p} + \frac{a}{p^2} + \frac{a^2}{p^3} + \ldots\right) \quad . \quad . \quad . \quad (758)$$

or

$$\frac{1}{p-a} = -\left(\frac{1}{a} + \frac{p}{a^2} + \frac{p^2}{a^3} + \ldots\right) \quad . \quad . \quad (759)$$

In a particular example (p. 168) it was found that the two different expansions gave particular integrals of distinctly different appearance which differed by a constant multiple of the complementary function. There was nothing to choose between the two expansions except convenience. From this point of view the expansion in equation 759 is generally far preferable. This is not, however, universally the case.

Solution of an equation such as equation 756 presupposes that $f(t)$ can be expressed as a continuous variation with t—that it can be represented by an equation. If $f(t)$ is discontinuous then $p \cdot f(t)$ is indeterminate at any discontinuity and the application of equation 759 by itself cannot supply the solution. Integration is, however, still possible and some hope of solution is offered by equation 758. It is this expansion which is used in applying Heaviside's methods.

In engineering work one is constantly concerned with discontinuous functions. At $t = 0$, the clutch of a car is engaged ;

what are the resulting forces and consequent motion? At $t = 0$, a telegraph key is depressed; what happens in the receiver? At $t = 0$, a lightning flash strikes a transmission line; what happens at the feeding point?

The most commonly occurring and simplest of such discontinuities may be represented by the *unit function*. At $t = 0$ a switch is closed. The voltage at the switching point which was previously zero suddenly becomes E and at that value it remains during the period of observation. The discontinuous unit function 1 is shown in Fig. 103. Fig. 104 shows a simple circuit. The voltage between A and B is E. The voltage between C and B is zero before the switch is closed and is E after. If the switch is closed at $t = 0$

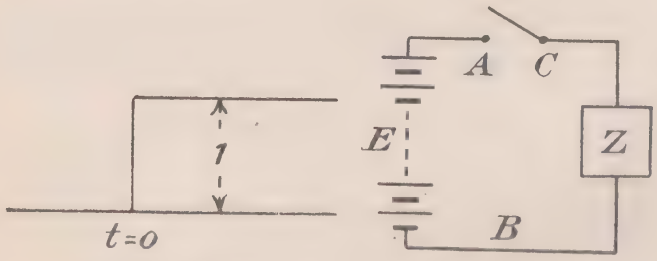


Fig. 103.

Fig. 104.

$$v_{AB} = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (760)$$

$$v_{CB} = E\,1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (761)$$

There is no need to describe v_{CB} further. The fact that it is zero before $t = 0$ and is E after is completely expressed in equation 761.* The particular case selected is merely illustrative. Clearly equation 761 with different symbols is of wider application.

If Z is of resistance R and inductance L the equation is $\varepsilon = iR + L\frac{di}{dt}$. But $\varepsilon = E\,1$ giving

$$E\,1 = iR + L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (762)$$

The operational equation is

$$i = \frac{E}{R + pL}\,1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (763)$$

Now the known solution to this equation is

$$i = \frac{E}{R}(1 - e^{-\frac{R}{L}t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (764)$$

* An alternative symbol for the unit fraction is $H(t)$. This has exactly the same significance as 1. On the same basis $H(t-t')$ expresses a function which is zero up to time t' and is thereafter unity.

Generalising this result equation 763 may be written as

$$i = \frac{E}{L} \cdot \frac{1}{p + \frac{R}{L}} 1$$

or putting $\frac{R}{L} = a$, $i = \frac{E}{L} \cdot \frac{1}{p + a} 1$

Again, the solution, equation 764, may be written

$$i = \frac{E}{L} \cdot \frac{L}{R}(1 - e^{-\frac{R}{L}t}) \text{ or}$$

$$i = \frac{E}{L} \cdot \frac{1 - e^{-at}}{a}$$

And hence

$$\frac{1}{p + a} 1 = \frac{1 - e^{-at}}{a} \quad . \quad . \quad . \quad . \quad (765)$$

or more strictly perhaps

$$\frac{1}{p + a} 1 = \frac{1 - e^{-at}}{a} 1$$

but it is more usual to omit the unit function in the right-hand member. It must be understood however that $\frac{1 - e^{-at}}{a}$ does not come into operation until $t = 0$.

The result expressed above has been obtained previously by operation upon zero for the complementary function and by operation upon unity for the particular integral. The expansion used was

$$\frac{1}{p + a} = \frac{1}{a} - \frac{p}{a^2} + \frac{p^2}{a^3} - \frac{p^3}{a^4} \quad . \quad . \quad . \quad . \quad (766)$$

Operating upon zero this gives Ce^{-at} as the complementary function and operating upon unity gives

$$\frac{1}{p + a} \cdot 1 = \frac{1}{a} \quad . \quad . \quad . \quad . \quad . \quad (767)$$

as the particular integral. Since at $t = 0$ the function is zero it follows that $C = -\frac{1}{a}$ and so

$$\frac{1}{p + a} 1 = \frac{1 - e^{-at}}{a}$$

Using Heaviside's method the result is obtained much more simply. From equation 350 (p. 175)

$$\frac{1}{p^n} \cdot 1 = \frac{t^n}{\lfloor n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (768)$$

$$\frac{1}{p+a} = \frac{1}{p} - \frac{a}{p^2} + \frac{a^2}{p^3} - \frac{a^3}{p^4} + \ldots \quad . \quad . \quad (769)$$

Operating with the right-hand member upon unity gives

$$\frac{1}{p+a} \cdot 1 = t - \frac{at^2}{\lfloor 2} + \frac{a^2t^3}{\lfloor 3} - \frac{a^3t^4}{\lfloor 4} \ldots = \frac{1 - e^{-at}}{a} \quad (770)$$

But this is zero when $t = 0$ and hence

$$\frac{1}{p+a} 1 = \frac{1 - e^{-at}}{a} 1 \quad . \quad . \quad . \quad . \quad . \quad (771)$$

The complementary function has become zero ($C = 0$). No constant of integration has to be determined; equation 771 is the complete solution. Further, equation 768 is complete when applied to unit function

$$\frac{1}{p^n} 1 = \frac{t^n}{\lfloor n} 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (772)$$

From this point equations 771 and 772 may be taken as expressing two pairs of equivalent operators. These equivalent operators may be readily extended. Writing $-a$ for a in equation 771 gives

$$\frac{1}{p-a} 1 = -\frac{1 - e^{at}}{a} 1 \quad . \quad . \quad . \quad . \quad (773)$$

Again $$\frac{p}{p+a} 1 = \left(1 - \frac{a}{p+a}\right) 1 = e^{-at} 1 \quad . \quad . \quad (774)$$

and $$\frac{p}{p-a} 1 = \left(1 + \frac{a}{p-a}\right) 1 = e^{at} 1 \quad . \quad . \quad (775)$$

The Expansion Theorem.

In the simple circuit consisting of an inductance and resistance in series the operational equation obtained was

$$i = \frac{E}{R + pL} 1$$

which might be written as

$$i = \frac{E}{f(p)} 1.$$

More complicated circuits will in general give rise to operational equations of the form

$$i = E\frac{f_1(p)}{f_2(p)}\,1 \quad . \quad . \quad . \quad . \quad . \quad (776)$$

Generally $f_1(p)$ is not of higher degree than $f_2(p)$, more usually it is of lower degree, and it is this case which will be particularly considered. Then $\frac{f_1(p)}{f_2(p)}$ may be expressed as

$$\frac{f_1(p)}{f_2(p)} = \frac{A}{p-a} + \frac{B}{p-b} + \frac{C}{p-c}, \text{ etc.} \quad . \quad . \quad (777)$$

where a, b, c, etc., are the roots of $f_2(p) = 0$. A, B, C, etc., may be found as follows.

$$\frac{f_1(p)}{f_2(p)} = \frac{f_1(p)}{(p-a)f_3(p)} = \frac{A}{p-a} + \frac{f_4(p)}{f_3(p)}$$

whence

$$\frac{f_1(p)}{f_3(p)} = A + (p-a)\frac{f_4(p)}{f_3(p)}$$

For $p = a$, $\frac{f_1(a)}{f_3(a)} = A$.

But $f_2(p) = (p-a)f_3(p)$.

Differentiating with respect to p gives

$$f_2'(p) = f_3(p) + (p-a)f_3'(p)$$

If $p = a$, $f_2'(a) = f_3(a)$

whence

$$A = \frac{f_1(a)}{f_2'(a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (778)$$

Similarly

$$B = \frac{f_1(b)}{f_2'(b)}, \text{ etc.}$$

so that equation 776 becomes

$$i = E\left(\frac{A}{p-a} + \frac{B}{p-b} + \frac{C}{p-c} + \ldots\right)1 \quad . \quad (779)$$

But from equation 773 this gives

$$i = E\left(-\frac{A}{a} - \frac{B}{b} - \frac{C}{c} \ldots\right)1$$

$$+ E\left(\frac{Ae^{at}}{a} + \frac{Be^{bt}}{b} + \frac{Ce^{ct}}{c} + \ldots\right)1 \quad . \quad . \quad (780)$$

But $\left(-\frac{A}{a}-\frac{B}{b}-\frac{C}{c}\ldots\right)$

$$=\left(\frac{A}{p-a}+\frac{B}{p-b}+\frac{C}{p-c}+\ldots\right)_{p=0}=\left[\frac{f_1(p)}{f_2(p)}\right]_{p=0}$$

$\left[\frac{f_1(p)}{f_2(p)}\right]_{p=0}$ may be written $\frac{f_1(0)}{f_2(0)}$

$$A=\left[\frac{f_1(p)}{f_2'(p)}\right]_{p=a}=\frac{f_1(a)}{f_2'(a)},\ \text{etc.}\quad \frac{A}{a}=\left[\frac{f_1(p)}{pf_2'(p)}\right]_{p=a},\ \text{etc.}$$

and equation 780 may be written

$$i=E\left[\frac{f_1(0)}{f_2(0)}+\sum_{a,b,c\ldots}\frac{f_1(p)e^{pt}}{pf_2'(p)}\right]1 \quad . \quad . \quad . \quad (781)$$

This is Heaviside's expansion theorem.

Example.

A resistance R, inductance L and capacitance C in series are connected at $t=0$ to a battery of e.m.f. E. Find the current if $R^2>4L/C$.

$$i=\frac{E}{R+pL+\frac{1}{pC}}1=\frac{Ep}{p^2L+pR+\frac{1}{C}}1$$

$$f_2(p)=p^2L+pR+\frac{1}{C}$$

Putting $f_2(p)=0$ gives $p=\dfrac{-R+\sqrt{R^2-4L/C}}{2L}=a$

and $$p=\frac{-R-\sqrt{R^2-4L/C}}{2L}=b$$

$$f_2'(p)=2pL+R,\ pf_2'(p)=2p^2L+pR$$

$$\frac{f_1(p)}{pf_2'(p)}=\frac{1}{R+2pL}$$

when $p=a$, $\dfrac{f_1(p)}{pf_2'(p)}=\dfrac{1}{\sqrt{R^2-4L/C}}$

when $p=b$, $\dfrac{f_1(p)}{pf_2'(p)}=-\dfrac{1}{\sqrt{R^2-4L/C}}$

$\frac{f_1(0)}{f_2(0)} = 0$ and so

$$i = \frac{E}{\sqrt{R^2 - 4L/C}}[e^{at} - e^{bt}]$$

where a and b have the values already given.

Simplification when $f_2(p)$ is a Product.

Some simplification of equation 781 is possible when $f_2(p)$ is a product. Suppose $f_2(p) = \phi(p) \cdot \psi(p)$

Then $$f_2'(p) = \phi(p) \cdot \psi'(p) + \phi'(p) \cdot \psi(p) \quad . \quad . \quad (782)$$

Suppose the roots of $\phi(p) = 0$ are a and b (two only are considered but the theorem is easily extended) and of $\psi(p) = 0$ are r and s. Then

$$\left.\begin{aligned} f_2'(a) &= \phi'(a) \cdot \psi(a) \\ f_2'(b) &= \phi'(b) \cdot \psi(b) \end{aligned}\right\} \text{ since } \phi(a) = 0 \text{ and } \phi(b) = 0$$

and

$$\left.\begin{aligned} f_2'(r) &= \phi(r) \cdot \psi'(r) \\ f_2'(s) &= \phi(s) \cdot \psi'(s) \end{aligned}\right\} \text{ since } \psi(r) = 0 \text{ and } \psi(s) = 0$$

and so

$$i = E\left[\frac{f_1(0)}{f_2(0)} + \frac{f_1(a)e^{at}}{a\phi'(a) \cdot \psi(a)} + \frac{f_1(b)e^{bt}}{b\phi'(b) \cdot \psi(b)} + \frac{f_1(r)e^{rt}}{r\phi(r) \cdot \psi'(r)} + \frac{f_1(s)e^{st}}{s\phi(s) \cdot \psi'(s)}\right] 1 \quad . \quad . \quad (783)$$

Variable Impressed Voltage.

When a voltage E of unit function form is impressed upon a circuit at $t = 0$ the current at any subsequent time has been seen to be given by

$$i = E\frac{f_1(p)}{f_2(p)} 1 \quad . \quad . \quad . \quad . \quad . \quad (776)$$

This is a function of t and may be written as

$$i = EU(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (784)$$

where $U(t)$ stands for the current at a time t after unit function voltage has been impressed upon the circuit.

If at $t = 0$ an e.m.f. E_0 is impressed, and at a subsequent time t_1 an additional e.m.f. E_1 is impressed, at a subsequent time t_2 a further e.m.f. E_2 is impressed, etc., then

$$i = E_0U(t) + E_1U(t - t_1) + E_2U(t - t_2), \text{ etc..} \quad (785)$$

Equation 785 may be readily extended to deal with a continuous variation of the impressed voltage. Thus the im-

pressed voltage, which may be denoted by $e = E(t)$, may be considered to be made up of an indefinite number of infinitesimal increments. In the succeeding time $\delta t'$ after a time t' the small increment of voltage is

$$\delta e = \frac{d}{dt'}[E(t')]\delta t' = E'(t')\delta t' \quad . \quad . \quad . \quad (786)$$

The consequent increment of current (at time t) is

$$\delta i = E'(t')\delta t' \,.\, U(t - t') \quad . \quad . \quad . \quad . \quad (787)$$

If at $t = 0$, $e = E(0)$ then the current at time t is, therefore

$$i = E(0)U(t) + \int_0^t E'(t') \,.\, U(t - t')dt' \quad . \quad . \quad (788)$$

An alternative form of this equation, usually simpler to apply in practice, is obtained by integrating by parts the second member of the right-hand side of this equation. Thus

$$\int_0^t E'(t') \,.\, U(t - t')dt'$$

$$= \Big[E(t') \,.\, U(t - t')\Big]_0^i - \int_0^t E(t') \,.\, \frac{d}{dt'}U(t - t')dt'$$

$$= E(t)U(0) - E(0) \,.\, U(t) + \int_0^t E(t')\frac{d}{d(t - t')}U(t - t')dt'$$

$$= E(t)U(0) - E(0) \,.\, U(t) + \int_0^t E(t')U'(t - t')dt'$$

Substituting in equation 788 gives

$$i = E(t)U(0) + \int_0^t E(t') \,.\, U'(t - t')dt' \quad . \quad . \quad (789)$$

This is Duhamel's superposition theorem applied to this particular problem. Equations 788 and 789 are alternative forms. The latter is usually found more convenient.

Example.

At $t = 0$ an e.m.f. $E \sin(\omega t + \alpha)$ is impressed upon a circuit of resistance R and inductance L. Find the current.

$$E(t) = E \sin(\omega t + \alpha)$$

$$E(t') = E \sin(\omega t' + \alpha)$$

$$U(t) = \frac{1}{R + pL}\,1 = \frac{1}{R}[1 - e^{-\frac{R}{L}t}]$$

$$U'(t) = \frac{1}{L}e^{-\frac{R}{L}}$$

$$U'(t-t') = \frac{1}{L}e^{-\frac{R}{L}(t-t')}$$

$$U(0) = 0$$

Hence using equation 789

$$i = E\sin(\omega t+\alpha)\,.\,0 + \int_0^t E\sin(\omega t'+\alpha)\,.\,\frac{1}{L}e^{-\frac{R}{L}(t-t')}dt'$$

or $$i = \frac{E}{L}e^{-\frac{R}{L}t}\int_0^t e^{\frac{R}{L}t'}\sin(\omega t'+\alpha)dt'$$

or $$i = \frac{E}{L}e^{-\frac{R}{L}t}\left[\frac{1}{\sqrt{\omega^2+\frac{R^2}{L^2}}}e^{\frac{R}{L}t'}\sin(\omega t'+\alpha-\phi)\right]_0^t$$

where $\tan\phi = \dfrac{\omega L}{R}$

$$i = \frac{E}{L}e^{-\frac{R}{L}t}\left[\frac{L}{\sqrt{\omega^2L^2+R^2}}e^{\frac{R}{L}t}\sin(\omega t+\alpha-\phi) - \frac{L}{\sqrt{\omega^2L^2+R^2}}\sin(\alpha-\phi)\right]$$

or $$i = \frac{E}{\sqrt{\omega^2L^2+R^2}}[\sin(\omega t+\alpha-\phi) - e^{-\frac{R}{L}t}\sin(\alpha-\phi)]$$

The Line Equations.

If the distance x be measured from the receiving end of the line, instead of from the sending end, the line equations 678 and 679 of Chapter XIX may be written

$$\frac{\partial v}{\partial x} = (R+pL)i$$

and

$$\frac{\partial i}{\partial x} = (G+pC)v$$

giving

$$\frac{\partial^2 v}{\partial x^2} = (R+pL)(G+pC)v \quad . \quad . \quad . \quad (790)$$

Writing $(R+pL)(G+pC) = -n^2$

where n is a function of p but not of x, equation 790 becomes

$$\frac{\partial^2 v}{\partial x^2} = -n^2 v$$

or

$$v = A\cos nx + B\sin nx \quad . \quad . \quad . \quad . \quad (791)$$

where A and B are determined by the terminal conditions. v is thus expressed as a function of p.

Writing $R + pL = Z$ and $G + pC = Y$ (note that Z and Y here are not the Z and Y of Chapter XIX, but are functions of p instead of $j\omega$) then the line equations become

$$\frac{\partial v}{\partial x} = Zi \quad . \quad . \quad . \quad . \quad . \quad . \quad (792)$$

$$\frac{\partial i}{\partial x} = Yv \quad . \quad . \quad . \quad . \quad . \quad . \quad (793)$$

and

$$i = \frac{1}{Z}\frac{\partial v}{\partial x} = \frac{n}{Z}(-A \sin nx + B \cos nx) \quad . \quad . \quad (794)$$

The Telegraph Equation.

It is assumed as a first approximation that the impedance of the receiving apparatus is negligible. That is at $x = 0$, $v = 0$ and therefore $A = 0$ (equation 791). At $x = l$, $v = E1 = B \sin nl\, 1$ or $B = \dfrac{E}{\sin nl}$ whence

$$v = E\frac{\sin nx}{\sin nl}1 \quad . \quad . \quad . \quad . \quad . \quad (795)$$

and

$$i = E\frac{n}{Z}\frac{\cos nx}{\sin nl}1 \quad . \quad . \quad . \quad . \quad . \quad (796)$$

Assuming the cable of Chapter XIX, having negligible inductance and leakance, $Z = R$ and $Y = pC$ and the equations become

$$\left.\begin{aligned} v &= E\frac{\sin nx}{\sin nl}1 \\ i &= E\frac{n}{R}\frac{\cos nx}{\sin nl}1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (797)$$

where $n^2 = -pCR$.

If to obtain the solutions for p one puts $\sin nl$ equal to zero one solution obtained is $nl = 0$. This solution is not acceptable without further examination. If $nl = 0$ then $\sin nx = 0$ and

$n \cos nx = 0$ and the solution becomes indeterminate. To overcome the difficulty the equations may be written as

$$\left.\begin{aligned} v &= E\frac{(\sin nx)/n}{(\sin nl)/n}\,1 \\ \text{and} \qquad i &= E\frac{1}{R}\,\frac{\cos nx}{(\sin nl)/n}\,1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (798)$$

Writing $(\sin nl)/n = 0$ gives

$$nl = \pm\pi,\ \pm 2\pi \ . \ . \ . \ \pm s\pi \ . \ . \ .$$
$$n^2l^2 = s^2\pi^2 = -pCRl^2$$

giving

$$p = -\frac{s^2\pi^2}{CRl^2} \quad . \quad . \quad . \quad . \quad . \quad (799)$$

s having all values from 1 to ∞.

(Note that the solution of $f_2(p) = 0$ is to obtain values of p, not of s; positive and negative values of s give the same value of p.)

In determining the terms of the expansion it is allowable to use equations 797 instead of 798. The same result is obtained in either case.

Writing $f_2(p) = \sin nl$, $f_2'(p) = l \cos nl \dfrac{dn}{dp}$.

But $n^2 = -pCR$ giving $2n\dfrac{dn}{dp} = -CR$.

Hence $$f_2'(p) = -\frac{CRl \cos nl}{2n}$$

and $$pf_2'(p) = \frac{nl \cos nl}{2}$$

For v one has

$$f_1(p) = \sin\frac{s\pi x}{l} \text{ and } \frac{f_1(0)}{f_2(0)} = \frac{x}{l},\ \frac{f_1(p)}{pf_2'(p)} = \frac{2 \sin\dfrac{s\pi x}{l}}{s\pi \cos s\pi}$$

whence $$v = E\left[\frac{x}{l} + \sum_{s=1}^{\infty} \frac{2 \sin\dfrac{s\pi x}{l} e^{-\frac{s^2\pi^2}{CRl^2}t}}{s\pi \cos s\pi}\right]$$

or

$$v = E\left[\frac{x}{l} + 2\sum_{s=1}^{\infty} (-1)^s \frac{\sin\dfrac{s\pi x}{l} e^{-\frac{s^2\pi^2}{CRl^2}t}}{s\pi}\right] \quad . \quad . \quad . \quad (800)$$

For i one has

$$f_1(p) = \frac{s\pi}{l}\cos\frac{s\pi x}{l} \text{ and } \frac{f_1(0)}{f_2(0)} = \frac{1}{l}$$

$$\frac{f_1(p)}{pf_2'(p)} = \frac{2\cos\dfrac{s\pi x}{l}}{l\cos s\pi}$$

and

$$i = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}(-1)^s \cos\frac{s\pi x}{l} e^{-\frac{s^2\pi^2}{CRl^2}t}\right] . \quad (801)$$

At the receiver, where $x = 0$

$$i_r = \frac{E}{Rl}\left[1 + 2\sum_{s=1}^{\infty}(-1)^s e^{-\frac{s^2\pi^2}{CRl^2}t}\right] . \quad (802)$$

With the necessary changes due to x being measured in the opposite direction these are the same equations as those derived in Chapter XIX.

Open-circuited Transmission Line having Negligible Losses.

The mechanism of wave transmission along a line is well illustrated by the line with no loss. Measuring x from the open end of the line the equations already given are

$$v = A\cos nx + B\sin nx . \quad (791)$$

and

$$i = \frac{n}{Z}(-A\sin nx + B\cos nx) . \quad (794)$$

If R and G are negligible $Z = pL$ and $n^2 = -p^2LC$. As the line is open at the receiving end $i = 0$ at $x = 0$ and therefore $B = 0$. It is assumed that a battery of e.m.f. E is switched at the sending end of the line at $t = 0$. If the length of the line is l then at $x = l$, $v = E\,1 = A\cos nl\,1$ or $A = \dfrac{E}{\cos nl}$. Hence the operational equations are

$$\left.\begin{aligned} v &= E\frac{\cos nx}{\cos nl}\,1 \\ i &= -E\frac{n\sin nx}{Z\cos nl}\,1 \end{aligned}\right\} . \quad (803)$$

The voltage equation only will be considered.

$$f_1(p) = \cos nx\,;\quad f_2(p) = \cos nl$$

Writing $f_2(p) = 0$ gives $nl = \pm (2s + 1)\frac{\pi}{2}$ where s has all integral values from 0 to ∞.

$f_2'(p) = -l \sin nl \frac{dn}{dp}$ but $2n\frac{dn}{dp} = -2pLC$ giving

$$f_2'(p) = \frac{pLC}{n} l \sin nl\,;\quad pf_2'(p) = \frac{p^2LCl \sin nl}{n} = -nl \sin nl$$

$$\frac{f_1(0)}{f_2(0)} = 1.$$

When s is even and $nl = +(2s + 1)\frac{\pi}{2}$,

$$\sin nl = 1,\ pf_2'(p) = -(2s + 1)\frac{\pi}{2}$$

When s is even and $nl = -(2s + 1)\frac{\pi}{2}$,

$$\sin nl = -1,\ pf_2'(p) = -(2s + 1)\frac{\pi}{2}$$

When s is odd and $nl = +(2s + 1)\frac{\pi}{2}$,

$$\sin nl = -1,\ pf_2'(p) = +(2s + 1)\frac{\pi}{2}$$

When s is odd and $nl = -(2s + 1)\frac{\pi}{2}$

$$\sin nl = 1,\ pf_2'(p) = +(2s + 1)\frac{\pi}{2}$$

$$p^2 = -\frac{n^2}{LC},\ n = \pm(2s + 1)\frac{\pi}{2l} \text{ whence } p = \pm j\frac{(2s + 1)}{l\sqrt{LC}} \cdot \frac{\pi}{2}$$

$$f_1(p) = \cos \frac{(2s + 1)\pi x}{2l}$$

When $s = 0$ the sum of the two terms obtained for $\frac{f_1(p)}{pf_2'(p)}e^{pt}$ is

$$-\frac{2}{\pi} \cos \frac{\pi x}{2l}\left[e^{j\frac{1}{l\sqrt{LC}} \cdot \frac{\pi}{2}t} + e^{-j\frac{1}{l\sqrt{LC}} \cdot \frac{\pi}{2}t}\right]$$

$$= -\frac{4}{\pi} \cos \frac{\pi x}{2l} \cos\left(\frac{\pi}{2l} \cdot \frac{1}{\sqrt{LC}}t\right)$$

When $s = 1$ the sum of the two terms obtained is

$$\frac{4}{3\pi} \cos \frac{3\pi x}{2l} \cos \left(\frac{3\pi}{2l} \cdot \frac{1}{\sqrt{LC}} t \right)$$

When $s = 2$ the sum of the two terms obtained is

$$-\frac{4}{5\pi} \cos \frac{5\pi x}{2l} \cos \left(\frac{5\pi}{2l} \cdot \frac{1}{\sqrt{LC}} t \right) \text{ and so on.}$$

Hence

$$v = E\left[1 - \frac{4}{\pi}\left\{ \cos \frac{\pi x}{2l} \cos \frac{\pi}{2l} \frac{1}{\sqrt{LC}} t - \frac{1}{3} \cos \frac{3\pi x}{2l} \cos \frac{3\pi}{2l} \frac{1}{\sqrt{LC}} t + \frac{1}{5} \cos \frac{5\pi x}{2l} \cos \frac{5\pi}{2l} \frac{1}{\sqrt{LC}} t \ldots \right\} \right]$$

or

$$v = E\left[1 - \frac{2}{\pi}\left\{ \cos \frac{\pi}{2l}\left(\frac{1}{\sqrt{LC}} t + x \right) - \frac{1}{3} \cos \frac{3\pi}{2l}\left(\frac{1}{\sqrt{LC}} t + x \right) + \frac{1}{5} \cos \frac{5\pi}{2l}\left(\frac{1}{\sqrt{LC}} t + x \right) \ldots \right\} - \frac{2}{\pi}\left\{ \cos \frac{\pi}{2l}\left(\frac{1}{\sqrt{LC}} t - x \right) - \frac{1}{3} \cos \frac{3\pi}{2l}\left(\frac{1}{\sqrt{LC}} t - x \right) + \frac{1}{5} \cos \frac{5\pi}{2l}\left(\frac{1}{\sqrt{LC}} t - x \right) \ldots \right\} \right] \quad (804)$$

Two wave trains are involved, one travelling to the left at a speed $u = \frac{1}{\sqrt{LC}}$ and the other travelling to the right at the same speed.

Writing

$$\theta_1 = \frac{\pi}{2l}(ut + x)$$

$$\theta_2 = \frac{\pi}{2l}(ut - x)$$

then

$$v = E\left[1 - \frac{2}{\pi}\left\{ \cos \theta_1 - \frac{1}{3} \cos 3\theta_1 + \frac{1}{5} \cos 5\theta_1 - \frac{1}{7} \cos 7\theta_1 \ldots \right\} - \frac{2}{\pi}\left\{ \cos \theta_2 - \frac{1}{3} \cos 3\theta_2 + \frac{1}{5} \cos 5\theta_2 - \frac{1}{7} \cos 7\theta_2 \ldots \right\} \right]$$

or

$$v = E[1 - f(\theta_1) - f(\theta_2)]$$

where $f(\theta) = \frac{2}{\pi}\left[\cos \theta - \frac{1}{3} \cos 3\theta + \frac{1}{5} \cos 5\theta - \frac{1}{7} \cos 7\theta \ldots \right]$

The value of this series has to be determined.

Let $C = \cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta \ldots$

and $jS = j\sin\theta - j\frac{1}{3}\sin 3\theta + j\frac{1}{5}\sin 5\theta \ldots$

then $C + jS = e^{j\theta} - \frac{1}{3}e^{3j\theta} + \frac{1}{5}e^{5j\theta} \ldots$

Now $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

whence $\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 \ldots$

Putting $x = jy$ gives

$$-j\frac{1}{2}\log\frac{1+jy}{1-jy} = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 \ldots$$

then

$$C + jS = -j\frac{1}{2}\log\frac{1+je^{j\theta}}{1-je^{j\theta}} = -j\frac{1}{2}\log\frac{1-\sin\theta+j\cos\theta}{1+\sin\theta-j\cos\theta}$$

$$= -j\frac{1}{2}\log\frac{j\cos\theta}{1+\sin\theta}$$

$1+\sin\theta$ must always be positive.

$\cos\theta$ may be positive or negative; it is positive when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; it is negative when $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

When $\cos\theta$ is positive

$$C + jS = -j\frac{1}{2}\left\{\log j + \log\frac{\cos\theta}{1+\sin\theta}\right\}$$

$$= -j\frac{1}{2}\left\{\log e^{j\frac{\pi}{2}} + \log\frac{\cos\theta}{1+\sin\theta}\right\}$$

whence $C = \frac{\pi}{4}$.

When $\cos\theta$ is negative

$$C + jS = -j\frac{1}{2}\left\{\log(-j) + \log\frac{-\cos\theta}{1+\sin\theta}\right\}$$

$$= -j\frac{1}{2}\left\{\log e^{-j\frac{\pi}{2}} + \log\frac{-\cos\theta}{1+\sin\theta}\right\}$$

whence $C = -\frac{\pi}{4}$.

Hence when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $f(\theta) = +\frac{1}{2}$

and when $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $f(\theta) = -\frac{1}{2}$

so that v at any point x has all the possible values given by $v = E\left[1 \pm \frac{1}{2} \pm \frac{1}{2}\right]$ namely 0, E and $2E$. It will be noted that since x cannot be greater than l, $\frac{\pi x}{2l}$ cannot be greater than $\frac{\pi}{2}$. Hence at $t = 0, f(\theta_1) = \frac{1}{2}, f(\theta_2) = \frac{1}{2}$ and $v = E\left[1 - \frac{1}{2} - \frac{1}{2}\right] = 0$. At $t = 0$, θ_1 is a positive angle less than $\frac{\pi}{2}$, θ_2 is a negative angle less than $\frac{\pi}{2}$. As time proceeds θ_1 and θ_2 grow in the positive direction. At any point x, v remains zero until θ_1 reaches $\frac{\pi}{2}$, that is when $ut_1 + x = l$ or $t_1 = \frac{l - x}{u}$. Then $f(\theta_1)$ suddenly changes from $+\frac{1}{2}$ to $-\frac{1}{2}$, $f(\theta_2)$ still remains $+\frac{1}{2}$. Hence the voltage suddenly rises to E. t_1 decreases with x and therefore all points nearer the sending end are at a potential E. t_1 is the time taken for the wave front to travel the distance $l - x$ from the sending end. The conditions on the line are represented by Fig. 105*a*. The wave of height E is travelling to the right with a speed u and has just reached the point x.

The next significant time is t_2 when θ_2 reaches $\frac{\pi}{2}$, given by

$$ut_2 - x = l \text{ or } t_2 = \frac{l + x}{u}$$

or the time that it takes for the wave front to travel from the sending end to the open end of the line, be reflected and have reached x on the return journey. At t_2, $f(\theta_2)$ suddenly changes from $+\frac{1}{2}$ to $-\frac{1}{2}$ and so the potential suddenly jumps from E to $2E$. The voltage has been doubled on reflection at the open end of the line. t_2 increases with x so that all points to the right of x are at a potential $2E$ and the step on the wave front is travelling to the left at a speed u (Fig. 105*b*).

No further change of potential occurs at x until θ_1 becomes $\frac{3\pi}{2}$. This occurs at $t_3 = \frac{3l - x}{u}$ when the wave front, reflected

at the sending point, is again travelling towards the right. At this instant $f(\theta_1)$ suddenly becomes $+\frac{1}{2}$ again, $f(\theta_2)$ still being $-\frac{1}{2}$ and the voltage at x suddenly falls to E again

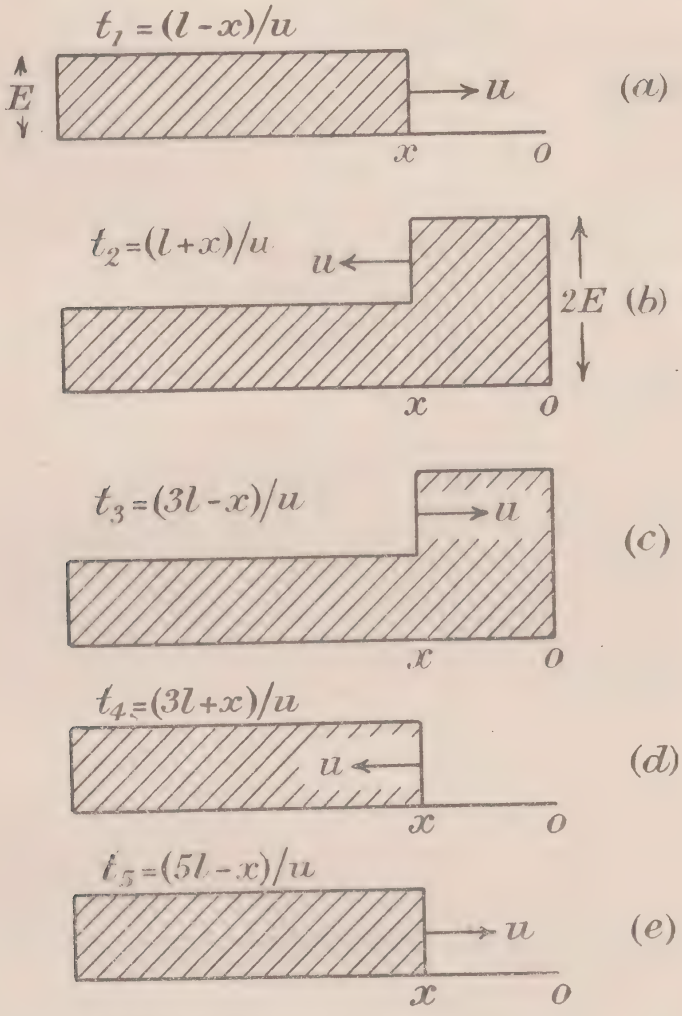


FIG. 105.

(Fig. 105*c*). The instantaneous distribution of voltage on the line is exactly the same at t_2 and t_3 but at t_2 the length of line at a potential $2E$ is increasing whilst at t_3 the length of line at a potential $2E$ is decreasing.

The next significant time is t_4 when θ_2 reaches $\frac{3\pi}{2}$. This occurs at $t_4 = \dfrac{3l+x}{u}$ when the wave front has been reflected at the open end of the line. Then $f(\theta_2)$ suddenly becomes $\frac{1}{2}$ and the voltage falls to zero. Conditions are shown in 105*d*. From this point the length of discharged line increases until the whole length is discharged

at $t = \frac{4l}{u}$. The next change at x occurs at $t_5 = \frac{5l - x}{u}$ when conditions are identical with those at t_1.

The sequence is tabulated below

t	$f(\theta_1)$	$f(\theta_2)$	$1 - f(\theta_1) - f(\theta_2)$	v
	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0
$t_1 = \frac{l - x}{u}$				
	$-\frac{1}{2}$	$+\frac{1}{2}$	1	E
$t_2 = \frac{l + x}{u}$				
	$-\frac{1}{2}$	$-\frac{1}{2}$	2	$2E$
$t_3 = \frac{3l - x}{u}$				
	$+\frac{1}{2}$	$-\frac{1}{2}$	1	E
$t_4 = \frac{3l + x}{u}$				
	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0
$t_5 = \frac{5l - x}{u}$				
	$-\frac{1}{2}$	$+\frac{1}{2}$	1	E
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Making use of the alternative expression for the unit function

$$f(\theta_1) = \tfrac{1}{2} - H(t - t_1) + H(t - t_3) - H(t - t_5) + \ldots$$
$$f(\theta_2) = \tfrac{1}{2} - H(t - t_2) + H(t - t_4) - H(t - t_6) + \ldots$$

whence

$$v = E[\{H(t - t_1) - H(t - t_3) + H(t - t_5) \ldots\} + \{H(t - t_2) - H(t - t_4) + H(t - t_6) \ldots\}]$$

Transmission Line having Resistance, Inductance, Leakance and Capacitance.

The solution is rather more lengthy if C, R, L and G all have finite values.

The general operational solution for the current is from equation 796

$$i = E\frac{n}{R + pL} \cdot \frac{\cos nx}{\sin nl} 1 \quad . \quad . \quad . \quad . \quad (805)$$

where $n^2 = -(R + pL)(G + pC)$.

$f_2(p) = (R + pL) \sin nl$

putting $\phi(p) = R + pL = 0$ gives one value of p namely $p = -\frac{R}{L}$. This corresponds to a in equation 783. The first part of the solution is

$$i_1 = E\frac{f_1(0)}{f_2(0)}$$

when $p = 0$, $n = j\sqrt{RG}$, $\cos nx = \cos j\sqrt{RG}x = \cosh \sqrt{RG}x$

$\sin nl = \sin j\sqrt{RG}l = j \sinh \sqrt{RG}l$

giving
$$i_1 = E\sqrt{\frac{G}{R}}\frac{\cosh \sqrt{RG}x}{\sinh \sqrt{RG}l} \quad . \quad . \quad . \quad . \quad (806)$$

The second part of the solution corresponds to the term $\frac{f_1(a)\varepsilon^{at}}{a\phi'(a)\psi(a)}$ of equation 783 where $\frac{f_1(a)}{\psi(a)} = \frac{n \cos nx}{\sin nl}$

When $\phi(p) = 0$ the solution is $p = -\frac{R}{L} = a$

then
$$n = \sqrt{0 \,.\, (G - CR/L)} = 0$$

and
$$\frac{n \cos nx}{\sin nl} = \frac{1}{l}$$

$\phi'(p) = L = \phi'(a)$; $a\phi'(a) = -R$ so that

$$i_2 = -\frac{E}{Rl}e^{-\frac{R}{L}t} \quad . \quad . \quad . \quad . \quad . \quad (807)$$

The third part of the solution corresponds to the sum of all terms of the form $\frac{f_1(r)\varepsilon^{rt}}{r\phi(r)\psi'(r)}$ of equation 783 where $\frac{f_1(r)}{\phi(r)} = \frac{n \cos nx}{R + pL}$

When $\psi(p) = \sin nl = 0, n = \frac{s\pi}{l}$ where s has all values from 1 to ∞.

$$\psi'(p) = l \cos nl \frac{dn}{dp}$$

but
$$n^2 = -(R + pL)(G + pC) \quad . \quad . \quad . \quad (808)$$

$$2n\frac{dn}{dp} = -(2pLC + LG + CR)$$

$$\psi'(p) = -\frac{l \cos nl(2pLC + LG + CR)}{2n}$$

Solving equation 808 for p gives $p = -\alpha \pm j\beta$ where

$$\alpha = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right) \text{ and } \beta = \sqrt{\frac{n^2}{LC} - \frac{1}{4}\left(\frac{R}{L} - \frac{G}{C}\right)^2}$$

$$p\psi'(p) = -\frac{LCl\cos nl}{2n}(2p^2 + 2\alpha p)$$

$$\frac{f_1(r)}{\phi(r)} = \frac{n\cos nx}{R + pL}$$

and so the coefficient of the general term is

$$-\frac{En^2\cos nx}{LCl\cos nl}\,\frac{1}{p(p+\alpha)(R+pL)}$$

p has the two values $-\alpha + j\beta$ and $-\alpha - j\beta$.

When p has the value $-\alpha + j\beta$

$$p(p+\alpha)(R+pL) = -\beta(\beta + j\alpha)R + j\beta L(\alpha - j\beta)^2$$

When p has the value $-\alpha - j\beta$

$$p(p+\alpha)(R+pL) = -\beta(\beta - j\alpha)R - j\beta L(\alpha + j\beta)^2$$

And so the general term becomes

$$-\frac{En^2\cos nx}{LCl\cos nl} \times \left[\frac{e^{(-\alpha+j\beta)t}}{-\beta(\beta+j\alpha)R + j\beta(\alpha-j\beta)^2L} + \frac{e^{(-\alpha-j\beta)t}}{-\beta(\beta-j\alpha)R - j\beta(\alpha+j\beta)^2L}\right]$$

$$= \frac{En^2\cos nx.e^{-\alpha t}}{\beta LCl\cos nl}\left[\frac{\cos\beta t + j\sin\beta t}{(\beta+j\alpha)R - j(\alpha-j\beta)^2L} + \frac{\cos\beta t - j\sin\beta t}{(\beta-j\alpha)R + j(\alpha+j\beta)^2L}\right]$$

$$= \frac{En^2\cos nx.e^{-\alpha t}}{\beta LCl\cos nl} \times \left[\frac{\cos\beta t(2\beta R - 4\alpha\beta L) + \sin\beta t\{2\alpha R - 2(\alpha^2-\beta^2)L\}}{(\alpha^2+\beta^2)R^2 + (\alpha^2+\beta^2)^2L^2 - 2LR\alpha(\alpha^2+\beta^2)}\right]$$

Substituting $\alpha^2 + \beta^2 = \dfrac{n^2 + RG}{LC}$,

$$\alpha^2 - \beta^2 = -\frac{n^2}{LC} + \frac{1}{2}\left(\frac{R^2}{L^2} + \frac{G^2}{C^2}\right)$$

and $R - 2\alpha L = -\frac{GL}{C}$ the general term becomes

$$= \frac{En^2 \cos nx.\varepsilon^{-\alpha t}}{\beta LCl \cos nl} \times \left[\frac{2\beta \cos \beta t\left(-\frac{GL}{C}\right) + 2 \sin \beta t\left\{\frac{n^2}{C} + \frac{GL}{2C}\left(\frac{R}{L} - \frac{G}{C}\right)\right\}}{\left(\frac{RG + n^2}{LC}\right)\left(\frac{L}{C}n^2\right)} \right]$$

$$= \frac{2E \cos nx.\varepsilon^{-\alpha t}}{l \cos nl}\left[\frac{-G \cos \beta t + \sin \beta t\left\{\frac{1}{\beta}\left[\frac{n^2}{L} + \frac{G}{2}\left(\frac{R}{L} - \frac{G}{C}\right)\right]\right\}}{RG + n^2} \right]$$

and $$i_3 = 2E \sum_{s=1}^{\infty} \frac{(-1)^s \cos \frac{s\pi x}{l} e^{-\alpha t} l}{s^2\pi^2 + RGl^2} \times \left[\frac{1}{\beta}\left\{\frac{s^2\pi^2}{l^2 L} + \frac{G}{2}\left(\frac{R}{L} - \frac{G}{C}\right)\right\} \sin \beta t - G \cos \beta t \right] \quad . \quad . \quad (809)$$

The complete solution is $i = i_1 + i_2 + i_3$ given by equations 806, 807 and 809.

The examples given are sufficient for the reader to obtain a general idea of Heaviside's method and to be able to deal with a large number of commonly occurring problems. Space does not permit pursuing the study further. Many applications of Heaviside's method will be found in works devoted particularly to the subject.*

* See, for instance, Berg's *Heaviside's Operational Calculus*.

CHAPTER XXII

CONJUGATE FUNCTIONS

When the line integral round any closed path in a vector field is zero, that is to say there is no curl, the field has a potential. The gradient of the potential is the vector. Electrostatic fields are of this nature, so are magnetic fields and many others. The chart of a field of this type consists of lines of force, or flux, and perpendicular to these lines are the equipotential surfaces. The analysis of fields possessing no

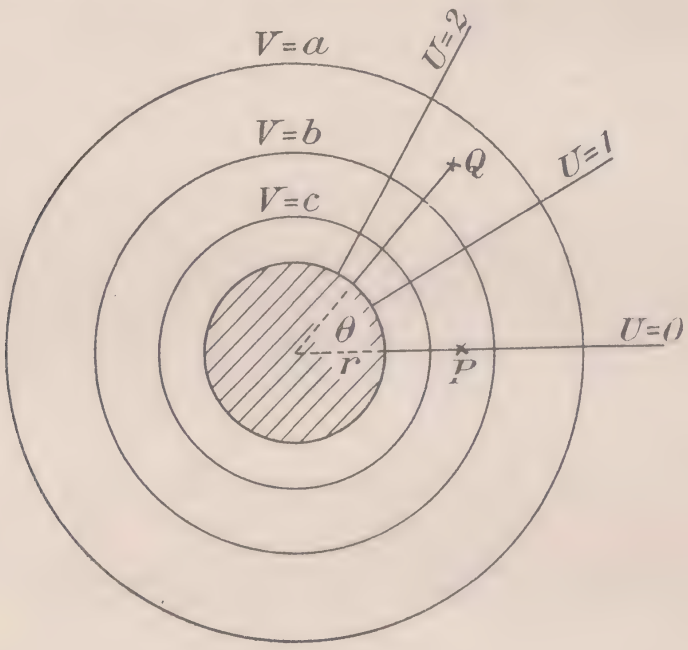


FIG. 106.

curl is much simpler than that of more general fields. In many practical cases a further simplification is afforded when the distribution of the field is two dimensional in character. Then the lines of force, or flux, are wholly in, or parallel to, say, the xy plane. When such a simplified distribution occurs, conjugate functions provide a very powerful method of analysis.

A simple case of a two-dimensional vector field is represented in Fig. 106. Many interpretations may be placed upon this figure but two particular interpretations will be considered :

(i) the hatched circle represents the plan of an indefinitely long positively charged cylinder, the surface density being σ. The lines of force stretch out radially. The circles perpendicular to these lines of force are plans of the equipotential surfaces defined by $V = a$, $V = b$, $V = c$, etc. Suppose that a line 1 cm. long be erected perpendicularly upon the paper at a point P on the line of force marked $U = 0$. Then if this line moves parallel to itself to some other point Q in the field it will cut through $\sigma r\theta$ Faraday tubes. If a function U be defined as

$$U = K \text{ (number of Faraday tubes cut)},$$

where K is some constant which may later be given a definite convenient value, then U and V together specify the point Q. As the unit perpendicular moves along a line of force it sweeps out a surface of constant U ; U only changes when tubes of force are cut, it then changes by an amount proportional to the number of tubes cut. Unless a conventional limitation is placed upon the path from P to Q, U might have any of the values $K\sigma r(2\pi n + \theta)$ where n is any integer. This ambiguity is resolved by imagining the surface $U = 0$ to be impenetrable ; $n = 0$ or the total angle θ is given by $0 \leq \theta < 2\pi$.

(ii) the hatched circle represents a perfectly conducting electrode placed upon a uniform and indefinitely extensive sheet of resistive material. Current flows from the electrode to the distant boundary. The equipotentials are the circles $V = a$, $V = b$, etc. The lines of current flow, or stream lines, are radial. U is now defined as

$$U = K' \text{ (current cut in passing from } P \text{ to } Q).$$

The same conventional limitation of θ holds.

U is termed the *stream function*.

Surfaces of constant U and surfaces of constant V are mutually prependicular.

Laplace's Equation.

For the two-dimensional distribution being considered Laplace's equation (equation 50, p. 45) reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (810)$$

This equation is satisfied by

$$V = f_1(x + jy) + f_2(x - jy) \quad . \quad . \quad . \quad (811)$$

No practical interpretation can be placed upon V unless it is wholly real and hence f_2 must be derivable from f_1 by changing jy into $-jy$. Then if $f_1 = u + jv$, $f_2 = u - jv$ and $V = 2u$.

Suppose a function S be defined as $S = -2v$ then

$$S + jV = 2j(u + jv) = 2jf_1(x + jy)$$

or

$$S + jV = \phi_1(x + jy) \quad . \quad . \quad . \quad . \quad . \quad . \quad (812)$$

Again defining $\phi_2(x + jy) = -j\phi_1(x + jy)$

$$\phi_2(x + jy) = -jS + V \quad . \quad . \quad . \quad . \quad (813)$$

Equation 812 shows that V may be expressed as the imaginary part of the most general arbitrary function of $x + jy$; it can also be expressed (equation 813) as the real part of the most general arbitrary function of $x + jy$. Either S or V are possible potentials. Two quantities S and V are said to be *conjugate functions* of x and y when $S + jV$ is a function of $x + jy$.

Meaning of S.

If the potential is V then it may be defined by

$$S + jV = \phi(x + jy) \quad . \quad . \quad . \quad . \quad (812a)$$

giving $$\frac{\partial S}{\partial x} + j\frac{\partial V}{\partial x} = \phi'(x + jy)$$

and $$\frac{\partial S}{\partial y} + j\frac{\partial V}{\partial y} = j\phi'(x + jy)$$

whence $$j\left(\frac{\partial S}{\partial x} + j\frac{\partial V}{\partial x}\right) = \frac{\partial S}{\partial y} + j\frac{\partial V}{\partial y}$$

Equating real and imaginary parts

$$\frac{\partial S}{\partial x} = \frac{\partial V}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (814)$$

and $$-\frac{\partial V}{\partial x} = \frac{\partial S}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (815)$$

These equations, known as the Cauchy-Riemann differential equations, define a fundamental property of the conjugate functions V and S. Consider two curves $V = a$, $S = b$ intersecting at the point P, the unit normals n_a and n_b making angles θ_a and θ_b with the axis of x (Fig. 107). If k_a is the rate

of change of V at P along n_a and k_b is the rate of change of S at P along n_b then

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= k_a \cos\theta_a, & \frac{\partial V}{\partial y} &= k_a \sin\theta_a \\ \frac{\partial S}{\partial x} &= k_b \cos\theta_b, & \frac{\partial S}{\partial y} &= k_b \cos\theta_b \end{aligned}\right\} \quad \ldots \quad (816)$$

Substituting from equation 814 gives

$$k_b \cos\theta_b = k_a \sin\theta_a$$

and substituting from equation 815 gives

$$k_b \sin\theta_b = -k_a \cos\theta_a$$

whence $k_a = k_b = k$, say

and $\tan\theta_a = -\cot\theta_b$

Hence $V = a$ and $S = b$ intersect at right angles. Lines of

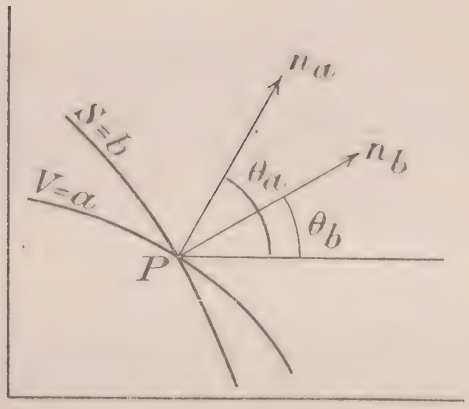


Fig. 107.

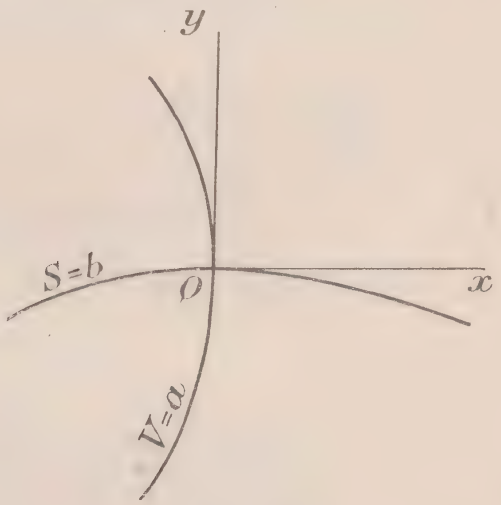


Fig. 108.

constant S are therefore lines of constant U. If $V = a$ is an equipotential then $S = b$ is a line of force (in the electrostatic case) or a stream line (in the case of steady current flow). If it can be shown that the change in S in moving from a point P to a point Q is equal to

K (number of Faraday tubes cut)

or K' (current cut)

(according to the alternative interpretations placed upon Fig. 106) then S may be identified with U. Consider the point of intersection O of an equipotential and a line of constant S. It has just been shown that these intersect at right angles. There is no loss of generality if the origin be transferred to O, the axis of y being tangent to the equipotential and the axis of x being tangent to the line of constant S (Fig. 108). Then

equations 816 become

$$\frac{\partial V}{\partial x} = k, \quad \frac{\partial V}{\partial y} = 0$$

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} = -k$$

But $\frac{\partial V}{\partial x} = -\varepsilon$ (equation 41, p. 41)

where ε is the electric force and $\varepsilon = \frac{4\pi D}{\kappa}$ (equation 53, p. 46), where D is the electric flux density at O and κ is the permittivity of the medium.

Hence $$\frac{\partial S}{\partial y} = \frac{4\pi D}{\kappa}$$

In moving a short distant δy along the equipotential the change in S is

$$\delta S = \delta y \frac{\partial S}{\partial y} = \frac{4\pi D}{\kappa}\delta y$$

$$= \frac{4\pi}{\kappa} \text{ (elementary number of Faraday tubes cut)}$$

and integrating one obtains

$$S_Q - S_P = \frac{4\pi}{\kappa}N$$

where N is the number of Faraday tubes cut in passing from P to Q. S may therefore be identified with U if K be given the value $\frac{4\pi}{\kappa}$. The symbol U, denoting the stream function, will be used hereafter instead of S.

Another interpretation of the stream function is provided by consideration of the current flow in a thin sheet of resistive material. Suppose the resistance of a strip of the material 1 cm. long and 1 cm. wide be ρ. Then in Fig. 108, $\frac{\partial V}{\partial x} = -X$, where X is the electric force along the axis of x. The current density in the sheet at the point O is $\frac{X}{\rho} = -\frac{1}{\rho}\frac{\partial V}{\partial x}$. If the current function (not stream function) at the origin is I then, at a distance δy from the origin it is

$$I + \delta I = I + \frac{X}{\rho}\delta y = I - \frac{1}{\rho}\frac{\partial V}{\partial x}\delta y$$

Hence $$-\frac{\partial V}{\partial x} = \rho\frac{\partial I}{\partial y} = \frac{\partial U}{\partial y}$$

so that $$U = \rho I$$

K' is therefore identified with ρ.

Making this substitution in equations 814 and 815 gives generally

$$-\frac{\partial V}{\partial x} = \rho\frac{\partial I}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (817)$$

and $$\rho\frac{\partial I}{\partial x} = \frac{\partial V}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (818)$$

When the point considered is not at the origin the conditions of equations 817 and 818 are represented by Fig. 109.

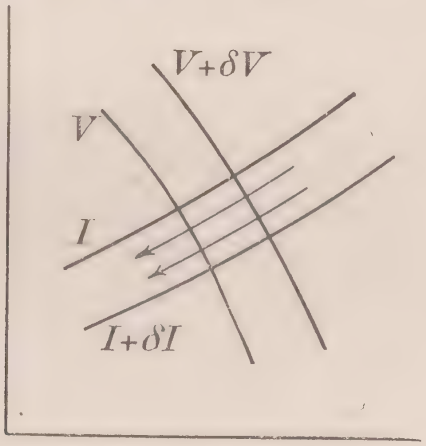


FIG. 109.

As drawn it will be seen that both $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ are positive, $\frac{\partial I}{\partial x}$ is positive, but $\frac{\partial I}{\partial y}$ is negative. The dissimilarity of sign in equations 817 and 818 (or in equations 814 and 815) is seen to be essential.

Conformal Representation.

The statement that V can be represented by either the real or the imaginary part of the most general arbitrary function of $x + jy$ will be more closely examined.

$$U + jV = \phi(x + jy)$$

Suppose the simplest possible interpretation of ϕ be considered namely

$$U + jV = x + jy \quad . \quad . \quad . \quad . \quad . \quad (819)$$

then $U = x$ and $v = y$.

Equipotential lines require y to be constant, they are parallel to the axis of x. The stream function is constant along lines parallel to the axis of y. This is clearly a possible condition. It occurs between the plates of a parallel plate condenser, except near its edges. The condition is represented in Fig. 110.

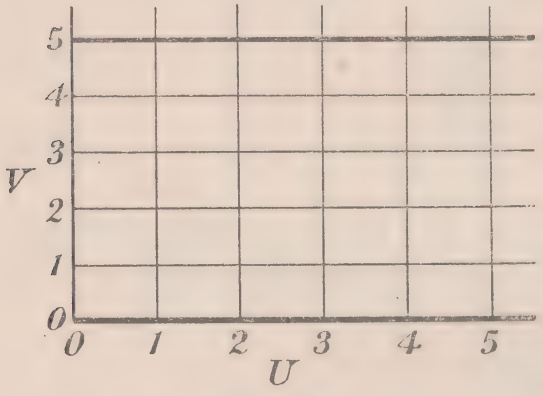


FIG. 110.

The equipotentials are parallel to the plates, represented in the figure by $y = 0$ and $y = 5$. The lines of force are perpendicular to the plates. Equally well the diagram can represent the field between two plates parallel to the axis of y when the equipotentials and the lines of constant U are interchanged. Or the diagram might represent the flow of current between two opposite sides of a rectangular plate, one set of lines being the equipotential lines, the other set the lines of flow.

If this diagram be transformed by giving a more general interpretation to ϕ, then another possible field distribution is obtained. Thus if

$$z = x + jy$$

and

$$W = U + jV$$

where

$$W = \phi(z),$$

values of z may be represented in one Argand diagram (in the z plane) and W in another (in the W plane). Then, if in one plane a possible field distribution be represented, the corresponding distribution in the other plane is also a possible field distribution. This important fact is the necessary interpretation of equation 812a. Examples of such transformations are given below. It will be found, and it is proved later that such must necessarily be the case, that to every elementary area in one plane there corresponds an exactly similarly shaped area in the other plane. The relative sizes and orientations of these elementary areas depend upon their position. From the fact that the shape of corresponding elements is the same in the two planes the process of transference from one plane to the other is known as *conformal representation.*

The Transformation $W = z^2$.

As a first example of conformal representation the transformation

$$W = z^2$$

is considered.

$$W = U + jV = (x + jy)^2 = x^2 - y^2 + 2jxy$$

$$U = x^2 - y^2 \text{ and } V = 2xy$$

To every point in the z plane there is a corresponding point in the W plane. Thus to the point $x = 2$, $y = 1$ corresponds the point $U = 3$, $V = 4$. If a point traverses any locus in

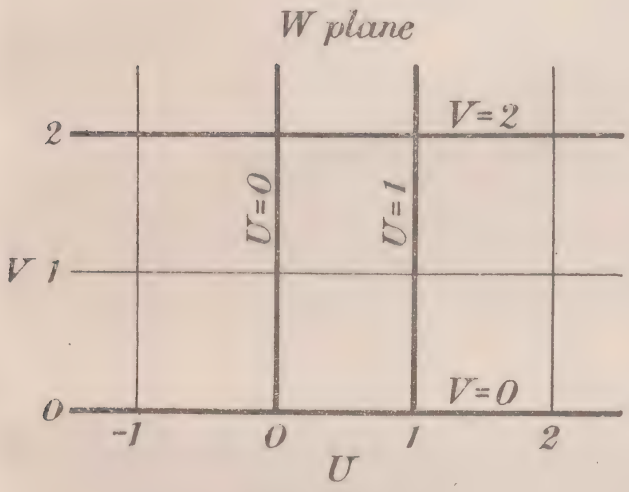


Fig. 111.

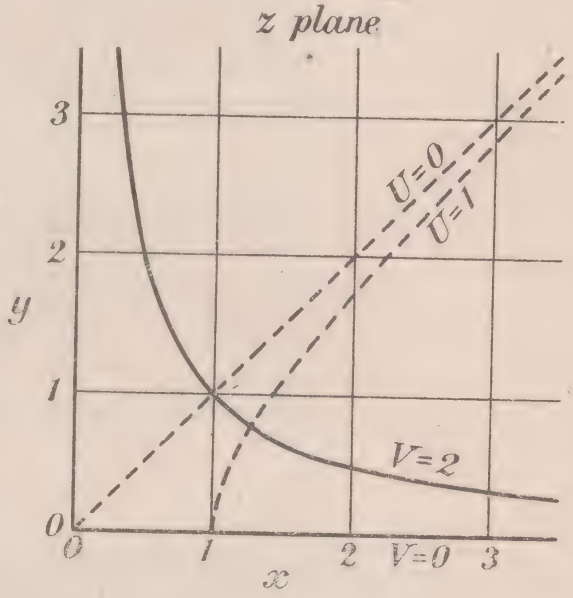


Fig. 112.

one plane, a corresponding point traverses a corresponding locus in the other plane. The case of interest arises when U and V in the W plane are constant. The corresponding curves in the z plane are

$$x^2 - y^2 = \text{constant}$$

and

$$2xy = \text{constant}.$$

These curves are families of hyperbolæ intersecting at right angles. Thus corresponding to $U = 0$, $x = y$ (Figs. 111 and 112). To $U = 1$ corresponds the hyperbola $x^2 - y^2 = 1$. To $V = 0$ corresponds $xy = 0$, or the axes of x and y. To $V = 2$ corresponds the hyperbola $xy = 1$. The general family of such curves in the z plane is given in Fig. 113. Of the various interpretations which may be placed upon Fig. 113 two may be given. If an extensive earthed sheet of metal be bent so that its plan coincides with the axes of x and y and

if another sheet, raised to a potential of 4 $kV.$, be bent so that its plan is represented by the curve $V = 4$, then the plan of the field distribution is given by Fig. 113. Again, if an extensive uniform rectangular sheet of resistive metal be fed by a perfectly

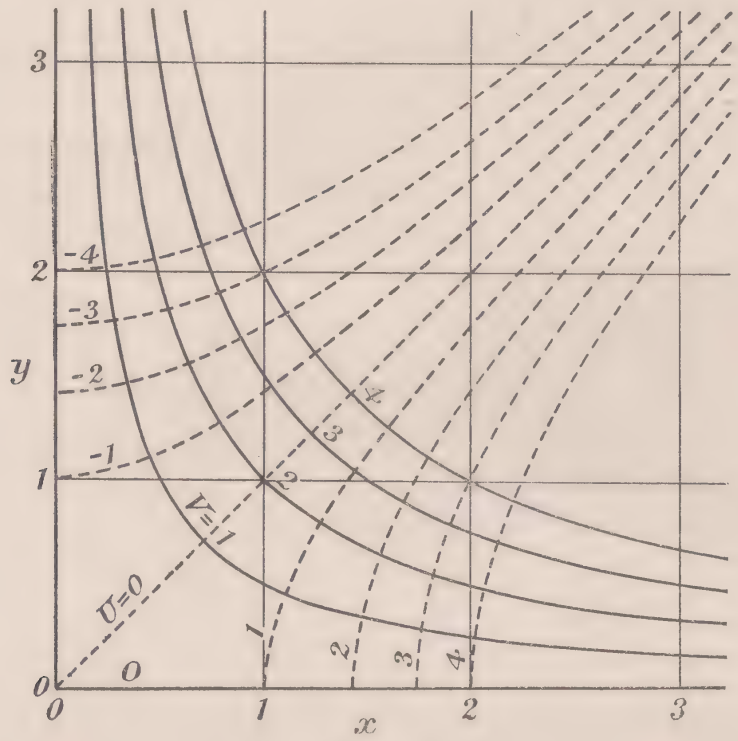


FIG. 113.

conducting electrode along the curve $V = 4$, the current flowing out along the two sides represented by the axes of x and y, then the equipotential lines and stream lines are represented by Fig. 113.

The Transformation $W = A \log z$.

The transformation

$$W = A \log z \quad . \quad . \quad . \quad . \quad . \quad (820)$$

is of particular importance. Substituting $z = re^{j\theta}$ gives

$$W = U + jV = A(\log r + j\theta)$$

$$U = A \log r, \; V = A\theta$$

The flux lines are circles and the equipotentials are planes $\theta =$ constant. In particular if conducting planes are placed at $\theta = 0$ and $\theta = \pi$ (separated of course by a minute distance to insulate them from one another) the field distribution is that shown in Fig. 114. If the plate at $\theta = 0$ is at zero potential the plate at $\theta = \pi$ is at a potential $A\pi$.

If U be interpreted as the potential instead of V the con-

centric circles represent the equipotentials and the radiating lines the lines of force. The transformation then corresponds to a line charge placed at the origin. Similarly the transformation $W = A \log (z - a)$ represents a line charge placed

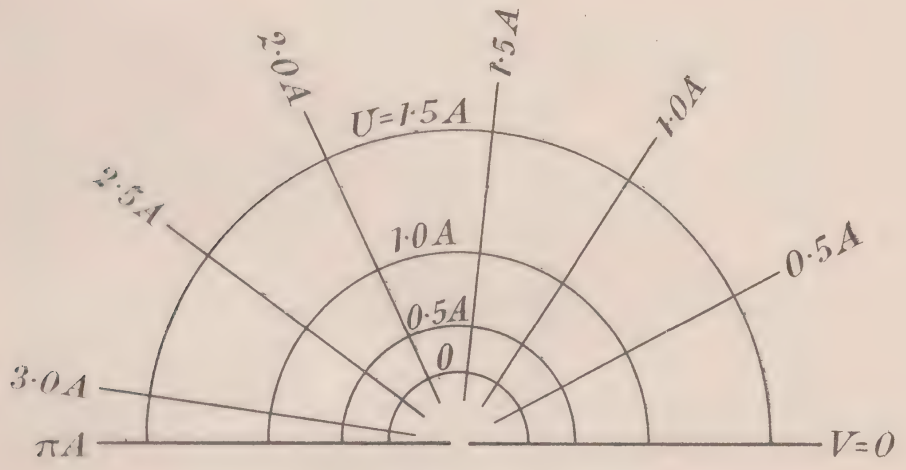


FIG. 114.

at $z = a$ and $W = A \log \dfrac{z - a}{z + a}$ a positive line charge placed at a and an equal negative line charge placed at $-a$. In this case if $x = 0$, $z = jy$ and $W = A \log \dfrac{jy - a}{jy + a} = A \log \dfrac{y + ja}{y - ja}$. This may be written

$$W = A \log \frac{\cos \phi + j \sin \phi}{\cos \phi - j \sin \phi} = A \log \varepsilon^{2j\phi} = 2jA\phi$$

where $\phi = \tan^{-1} \dfrac{a}{y}$. The potential U at $x = 0$ is zero and hence $W = A \log \dfrac{z - a}{z + a}$ gives the transformation for a line charge placed at a distance a from a plane at earth potential.

The Transformation $W = A \log z$ (*continued*).

A case of some interest in view of subsequent work is the representation in the W plane of a rectangular system in the z plane.

$$U = A \log r = A \log \sqrt{x^2 + y^2}$$

$$V = A\theta = A \tan^{-1} \frac{y}{x}$$

In Fig. 115 is shown a rectangular system in the z plane. Fig. 116 is the transformation of this system into the W plane (A being assumed to be unity). Interest centres particularly in lines of constant y. As y diminishes the points for which

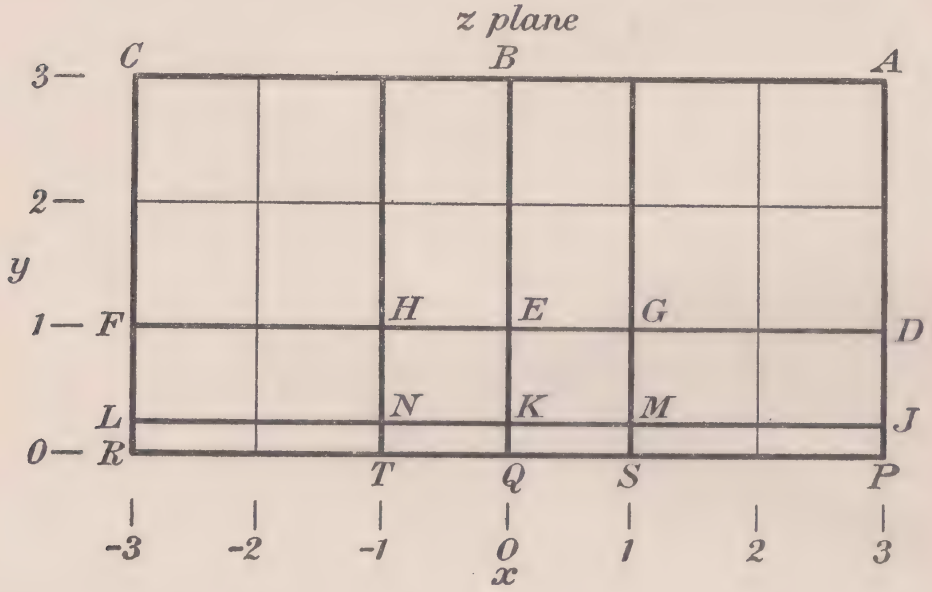


FIG. 115.

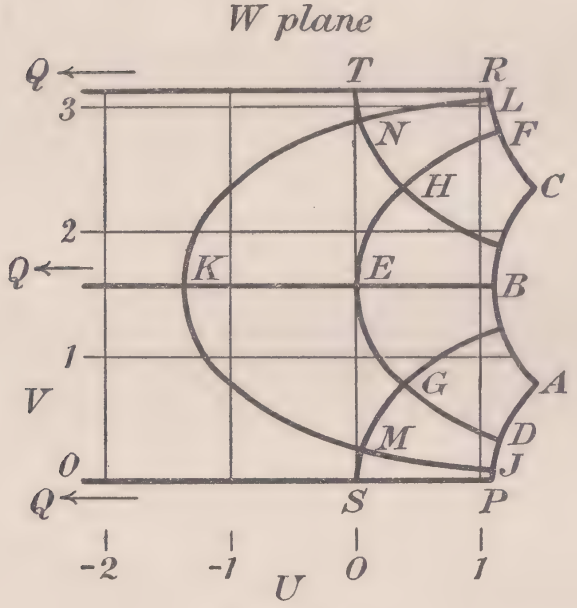


FIG. 116.

$x = 0$, namely B, E, K and Q, move towards the left in Fig. 116. Except for the point Q (when V is indeterminate) the value of V is constant at $\frac{\pi}{2}$; U becomes increasingly negative as y decreases, finally becoming $-\infty$ when $y = 0$. The line

PSQTR of Fig. 115 breaks up in Fig. 116 into two semi-infinite parallel portions separated by a vertical distance π. For all points to the right of Q in Fig. 115, $V = 0$ and for all points to the left $V = \pi$.

Ratio of Transformation.

It has already been mentioned (p. 334) that to any infinitesimal area in the z plane here corresponds an exactly similar shaped area in the W plane. The relative sizes and orientations of these small areas depend upon the co-ordinates and the form of the transformation. This will be examined further, first for the transformation $W = z^2$. In the z plane of Fig. 117 two small squares are shown, of the same size. Their transformations to the W plane are given in Fig. 118. The relations

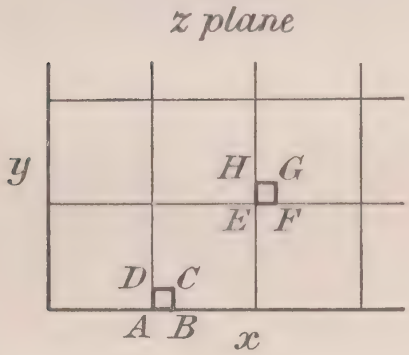


Fig. 117.

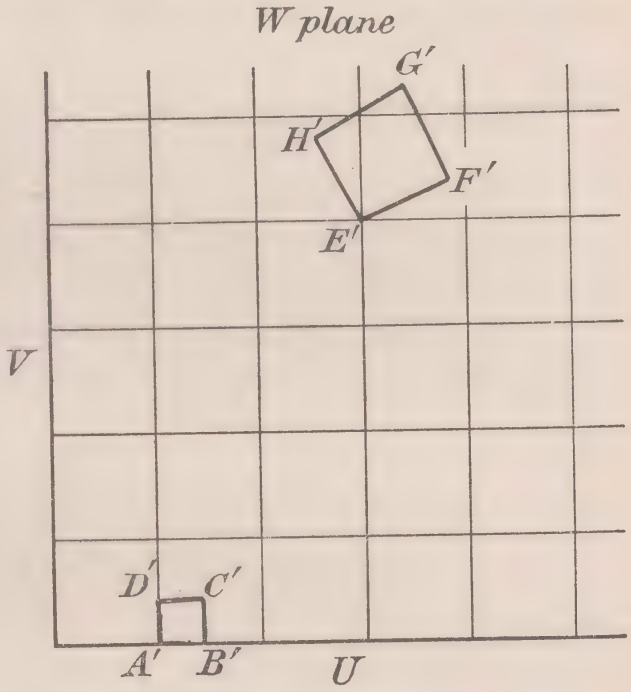


Fig. 118.

existing are $U = x^2 - y^2$ and $V = 2xy$. Thus to A, having co-ordinates $x = 1$, $y = 0$, corresponds the point A' having co-ordinates $U = 1$, $V = 0$. Similarly to B, having co-ordinates $x = 1{\cdot}2$, $y = 0$, corresponds B' having co-ordinates $U = 1{\cdot}44$, $V = 0$. The co-ordinates are given completely in the Table on page 340. It is seen that the figures $A'B'C'D'$ and $E'F'G'H'$ are approximately squares. They would be exact squares were the squares $ABCD$ and $EFGH$ indefinitely small. In being transferred to the W plane their size has been changed and in a ratio depending upon their distances from the origin. Further, the square $EFGH$ has been changed in orientation.

	x	y	U	V	
A	1	0	1	0	A'
B	1·2	0	1·44	0	B'
C	1·2	0·2	1·40	0·48	C'
D	1	0·2	0·96	0·40	D'
E	2	1	3	4	E'
F	2·2	1	3·84	4·4	F'
G	2·2	1·2	3·40	5·28	G'
H	2	1·2	2·56	4·8	H'

Another transformation will be considered, namely

$$W = j \log 20z$$

Writing $z = x + jy = r(\cos\theta + j \sin\theta) = re^{j\theta}$

$$W = U + jV = j(j\theta + \log 20r)$$

$$U = -\theta, \quad V = \log 20r$$

In the W plane lines of constant potential and constant stream function are represented by straight lines parallel to the axes (Fig. 120). In the z plane the equipotentials are circles and the stream lines are radii (Fig. 119). Two small

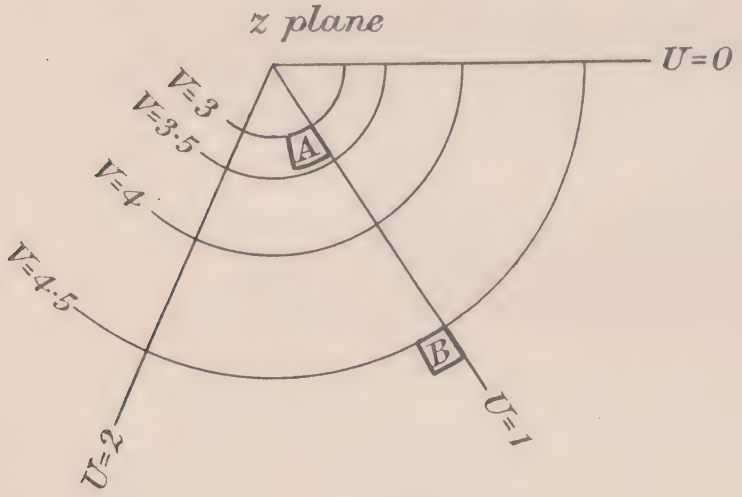


Fig. 119.

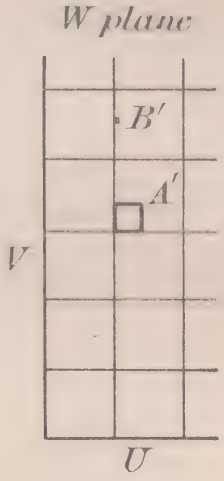


Fig. 120.

and rather similar areas A and B in the z plane of approximately the same size are shown in Fig. 119. A' and B' (Fig. 120) are the corresponding areas in the W plane. The difference in the ratio of transformation will be noted. The ratio increases with increase in the curvature of the equipotential surfaces in the z plane.

The value of this transformation ratio may now be determined. Corresponding to any point P in the z plane there is a point P' in the W plane, the respective co-ordinates being z and W. If Q be another point in the z plane separated from P by a very small amount δz then Q' in the W plane will be separated from P' by a very small amount δW. The linear transformation ratio is

$$\underset{\delta z \to 0}{\text{Lt.}} \frac{\delta W}{\delta z} = \frac{dW}{dz} = \frac{d}{dz}\phi(z) = \phi'(z)$$

But $\dfrac{\partial W}{\partial x} = \phi'(z)$ and hence

$$\frac{dW}{dz} = \frac{\partial W}{\partial x} = \frac{\partial}{\partial x}(U + jV) = \frac{\partial U}{\partial x} + j\frac{\partial V}{\partial x}$$

But from equation 814

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

And hence

$$\frac{dW}{dz} = \frac{\partial V}{\partial y} + j\frac{\partial V}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (821)$$

But $\dfrac{\partial V}{\partial x} = -X$ and $\dfrac{\partial V}{\partial y} = -Y$ where X and Y are the components of the electric force ε parallel respectively to the axes of x and y. Hence

$$\frac{dW}{dz} = -\varepsilon(\cos\alpha + j\sin\alpha) \quad . \quad . \quad . \quad (822)$$

where

$$\alpha = \tan^{-1}\frac{X}{Y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (823)$$

It is seen that the ratio is, in general, complex ; the element δz is multiplied by the factor $-\varepsilon$ in transferring it from the z to the W plane and it is rotated through an angle α given by equation 823. This complex ratio depends only upon the *position* of the element δz and does not depend upon the magnitude (so long as it is small) or the direction of δz. It therefore follows that each elementary area in the z plane, on transference to the W plane, though changed in size and orientation, preserves its shape. A circle remains a circle, a square remains a square. This is only true when the areas are indefinitely small. The transformation ratio varies from point to point

and, as the size of a square in the z plane increases, the departure from squareness of its corresponding figure in the W plane increases. The converse of this is seen in Fig. 113. The mesh there shown is composed of figures which individually do not depart greatly from squares. Considerable departure from squareness results when a large figure is viewed. Relevant

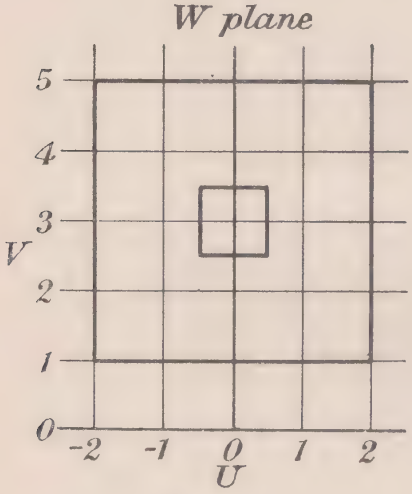


Fig. 121*a*.

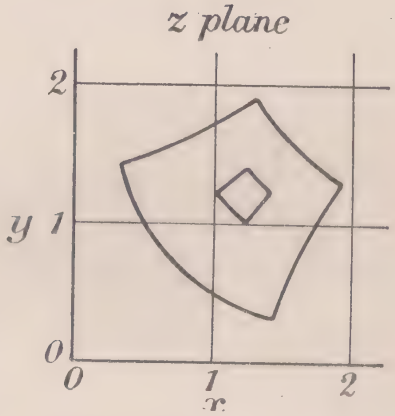


Fig. 121*b*.

parts of this diagram are given in Fig. 121*b*, together with the corresponding areas in the W plane in Fig. 121*a*. It is seen that the small square in the W plane is represented by a small figure which is nearly a square in the z plane ; the larger square in the W plane is represented by a figure in the z plane which is far from being a square.

Differences between the figures in the z and W planes are entirely due to the variation of the transformation ratio $\frac{dW}{dz}$. The figure in which interest is particularly centred is a mesh in the z plane which transforms into a uniform square mesh in the W plane. Then since the mesh in the W plane represents a possible distribution of equipotentials and flux lines it follows (equations 812 and 813) that the mesh in the z plane also represents a possible field distribution. It has already been seen that from a given transformation the field distribution can be readily determined. Practice presents the converse problem. Certain equipotentials are known ; it is required to determine the transformation fulfilling the known conditions. Then the remaining field distribution can be determined. To determine the required transformation is often far from easy.

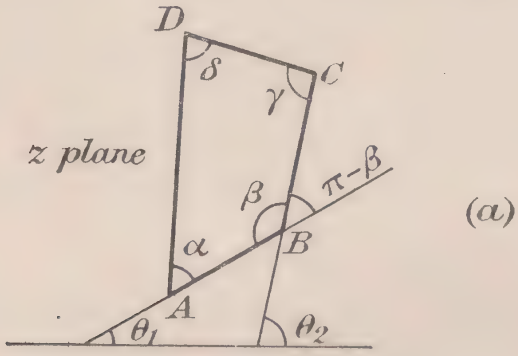
A transformation which has already been considered, namely $W = z^2$, may be usefully examined further. The transformation ratio is $\frac{dW}{dz} = 2z$. When $z = 1$, $\frac{dW}{dz} = 2$. A small figure at the position $x = 1$, $y = 0$ on transference to the W plane has its linear dimensions doubled and there is no change of orientation. It will be seen that this agrees with Figs. 117 and 118—the square $ABCD$ transforms to $A'B'C'D'$ of twice the dimensions. Both squares have the same orientation. If $z = 2 + j$ then $\frac{dW}{dz} = 4 + 2j$; on transference from the z plane to the W plane a square has its side increased $\sqrt{4^2 + 2^2}$, or 4·47, times and it is tilted through an angle $\tan^{-1} 0{\cdot}5$. It is seen that this also agrees with Figs. 117 and 118.

As the origin is approached the ratio of transformation $\frac{dW}{dz} = 2z$ reduces, becoming zero at the origin itself. But the origin is a point on the equipotential surface formed by the two axes (Fig. 113). The curvature of the equipotential surfaces increases as the origin is approached; at the origin itself it becomes infinitely concave. It is well known that at a re-entrant corner the electric force is zero; the electric force is numerically equal to the ratio of transformation (equation 822). The converse is seen in Fig. 119 where the equipotentials are convex away from the origin. In this case as the origin is approached the convex curvature increases, becoming infinite at the origin itself. The limit of convex curvature is reached at an edge where it is known that the electric force becomes infinite. It is seen therefore that if an equipotential surface is polygonal then, at a corner, $\frac{dW}{dz}$ is zero or infinite according as the corner is convex or concave towards the origin.

The Schwarz-Christoffel Transformation.

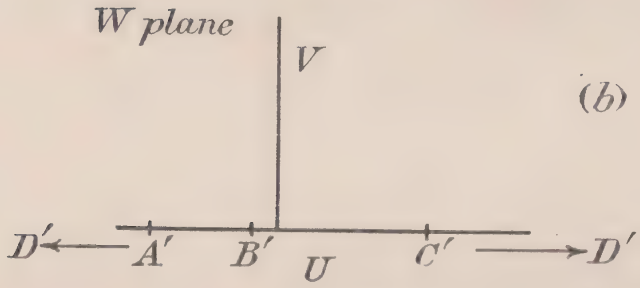
Schwarz and Christoffel have given a transformation enabling a linear polygon in the z plane to be opened out into a straight line in the W plane. This straight line may conveniently be the equipotential $V = 0$, stretching from $U = -\infty$ to $U = +\infty$. One point in the polygon is at both $+\infty$ and $-\infty$; often this may be conveniently arranged to be one of

the corners of the polygon. Thus the polygon $ABCD$ in the plane of z (Fig. 122a) is to correspond with the line $V = 0$ or $A'B'C'D'$ in the plane of W (Fig. 122b), the values of U at A', B', C', D' being respectively a, b, c, d. Since $V = 0$ the complete co-ordinates, W, of A', B', C', D' are also a, b, c, d.



(a)

FIG. 122a.



(b)

FIG. 122b.

Now when any element is transferred from the z plane to the W plane it is multiplied by the complex quantity $\frac{dW}{dz}$. Conversely an element may be transferred from the W plane to the z plane on multiplying it by the complex quantity $\frac{dz}{dW}$. The effect of multiplying an element by $\frac{dz}{dW}$ is to increase its magnitude mod. $\left(\frac{dz}{dW}\right)$ times, at the same time rotating the element anticlockwise through an angle arg. $\left(\frac{dz}{dW}\right)$.

At the corners of the figure $ABCD$, $\frac{dz}{dW}$ must be either infinite or zero (since $\frac{dW}{dz}$ is either zero or infinite). Consider the line AB as made up of a number of elements of the

line $A'B'$ multiplied by the complex and variable multiplier $\frac{dz}{dW}$. Each of these elements of $A'B'$ must be rotated through the same angle θ_1, the angle which AB makes with the axis of x.

Now $\frac{dz}{dW}$ will be zero or infinite at the point B if $\frac{dz}{dW}$ is a product of factors of which one is $(W-b)^{\mu_b}$ for, at the point B', $(W-b)$ is zero and $(W-b)^{\mu_b}$ is zero or infinite according as μ_b is positive or negative. The condition that $\frac{dz}{dW}$ is zero or infinite at every corner is therefore satisfied by a relation of the form

$$\frac{dz}{dW} = K(W-a)^{\mu_a}(W-b)^{\mu_b}(W-c)^{\mu_c}(W-d)^{\mu_d} \quad (824)$$

The nature of K is for the moment undetermined, but if it be imagined to be a numerical constant it will be seen that the other necessary condition is also satisfied. Passing along the side AB, $V=0$ and $a<U<b$. The quantities $W-a$, $W-b$, $W-c$, $W-d$ are all real quantities positive or negative. The argument of each positive quantity is zero; the argument of each negative quantity is $\pm\pi$. As one passes round the figure in the z plane the argument of δz remains constant along one side, suddenly changing as a corner is passed. Similarly as one passes along the figure in the W plane the signs of $W-a$, $W-b$, etc., remain constant, except when one of the points A', B', C', etc., is passed. When for instance B' is passed, $W-b$ changes sign, that is to say its argument changes by $\pm\pi$. The necessary condition is that along any one side of the figure the argument of $\frac{dz}{dW}$ must not change. From equation 824, if K is a constant

$$\arg\left(\frac{dz}{dW}\right) = \mu_a \arg(W-a) + \mu_b \arg(W-b) + \mu_c \arg(W-c) + \mu_d \arg(W-d) \quad . \quad . \quad (825)$$

and it is seen that this necessary condition is satisfied.

Along the side AB, $\arg\left(\frac{dz}{dW}\right) = \theta_1$; along the side BC, $\arg\left(\frac{dz}{dW}\right) = \theta_2$. As the corner B is passed $\arg\left(\frac{dz}{dW}\right)$ sud-

denly changes by $\theta_2 - \theta_1 = \pi - \beta$. At the same time arg $(W - b)$ changes by π. Hence

$$\pi - \beta = \pm \mu_b \pi \quad . \quad . \quad . \quad . \quad . \quad (826)$$

The sign may be determined from physical considerations or by reference to previous work. It is assumed that the field considered is that *within* the polygon. Then if the internal angle is less than π, $\frac{dz}{dW}$ is infinite (see, for instance, Fig. 113) ; if the internal angle is greater than π then $\frac{dz}{dW}$ is zero. At B the relevant factor is $(W - b)^{\mu_b}$. μ_b is positive if $\beta > \pi$; μ_b is negative if $\beta < \pi$. Hence for the field within the polygon the sign to be taken in equation 826 is the negative one giving

$$\mu_b = \frac{\beta}{\pi} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (827)$$

Similarly $\mu_a = \frac{\alpha}{\pi} - 1$; $\mu_c = \frac{\gamma}{\pi} - 1$, etc., and hence the required transformation is

$$\frac{dz}{dW} = K(W-a)^{\frac{\alpha}{\pi}-1}\,(W-b)^{\frac{\beta}{\pi}-1}\,(W-c)^{\frac{\gamma}{\pi}-1}\,(W-d)^{\frac{\delta}{\pi}-1} \quad (828)$$

Successive Transformations.

Transformations may be made successively. For instance a transformation may be effected from the z plane to the ζ plane where $\zeta = \phi(z)$. A second transformation may then follow

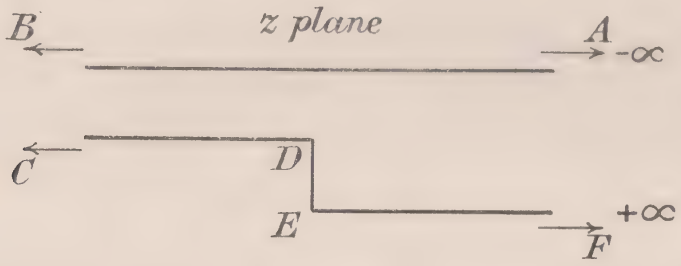


FIG. 123.

from the ζ plane to the W plane according to $W = f(\zeta)$. W is thus a function of z. One application of this double transformation may be illustrated. Suppose it is desired to determine the field distribution between two plates of the form indicated in Fig. 123 when a steady difference of potential is established between them. The figure is treated as a polygon,

the points B and C being coincident at an infinite distance. By means of the Schwarz-Christoffel transformation the corresponding figure in the plane of ζ is formed, the point BC being conveniently at the origin and the whole figure coinciding

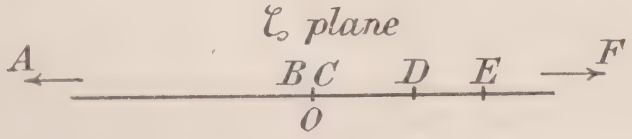


FIG. 124.

with $\eta = 0$ where $\zeta = \xi + j\eta$ (Fig. 124). Then the transformation of equation 820 is applied as

$$W = A \log \zeta$$

giving a difference of potential between the plates AB and $CDEF$ of (see p. 336).

$$V_1 = A\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (829)$$

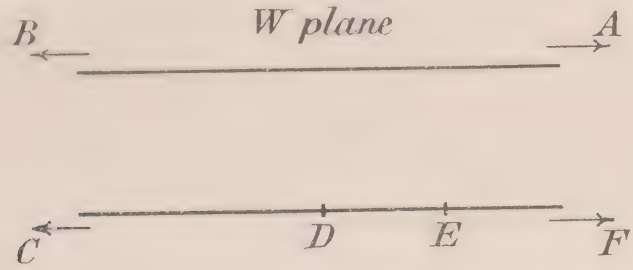


FIG. 125.

Fig. 125 represents this transformation in the W plane. The distribution in this plane is defined by

$$W = U + jV,$$

equipotentials and flux lines giving a simple rectangular mesh. Reversing the process the distribution in the z plane is obtained. A case very similar to this will be worked out.

Distribution of Current and Potential in a Conductor of Uniform Thickness whose Breadth Suddenly Changes.

This problem was first worked out by Professor C. H. Lees in 1908.* Alternative methods of solving the integral involved are here given, one substantially that given in Professor Lees' paper, the other substantially that given by Professor Miles Walker.†

* *Proceedings of the Physical Society*, Vol. XXI, p. 734.

† *Conjugate Functions for Engineers*, Oxford Univ. Press, 1933.

The form of the conductor is shown in Fig. 126. It consists of a long bar whose width over a considerable length is c. The width is suddenly reduced to b, there being a considerable length at this reduced width. It is desired to determine the current and potential distribution in the neighbourhood of the change of section. The origin in the plane of z is chosen at D. The origin in the ζ plane corresponds to the convergence of B and C, B being coincident with the negative side of

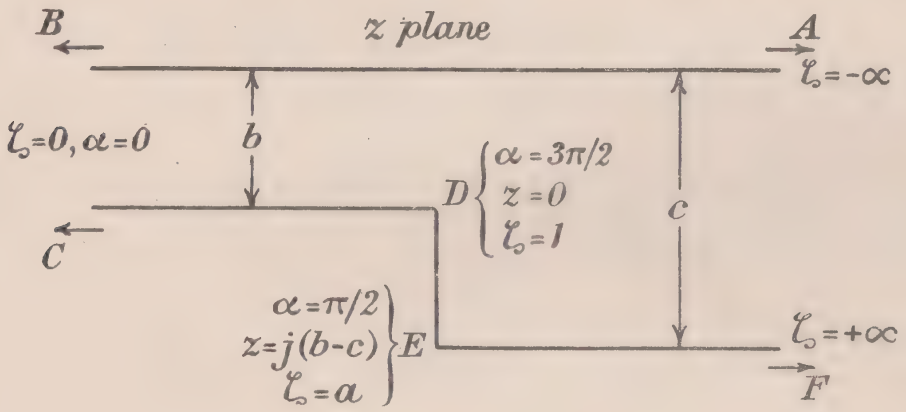


Fig. 126.

the origin and C with the positive side. At D, ζ is chosen as unity (this is only a matter of scale ; its great convenience, as will be seen later, is due to the fact that the logarithm of unity is zero and D will be at the origin in the W plane as well as in the z plane). The z co-ordinate of E is $j(b-c)$; its ζ co-ordinate, at present undetermined, is assigned the value a. The internal angles α of the polygon in the z plane are : at BC, 0 ; at D, $\frac{3\pi}{2}$; at E, $\frac{\pi}{2}$. Hence at BC, $\frac{\alpha}{\pi} - 1 = -1$ and $\zeta = 0$. At D, $\frac{\alpha}{\pi} - 1 = \frac{1}{2}$ and $\zeta = 1$. At E, $\frac{\alpha}{\pi} - 1 = -\frac{1}{2}$ and $\zeta = a$. Hence the Schwarzian equation is

$$\frac{dz}{d\zeta} = K\zeta^{-1}(\zeta - 1)^{\frac{1}{2}}(\zeta - a)^{-\frac{1}{2}} \quad . \quad . \quad . \quad (830)$$

or

$$\frac{dz}{d\zeta} = K\frac{1}{\zeta}\sqrt{\frac{\zeta - 1}{\zeta - a}} \quad . \quad . \quad . \quad . \quad . \quad (831)$$

The transformation to the ζ plane is given in Fig. 127. The

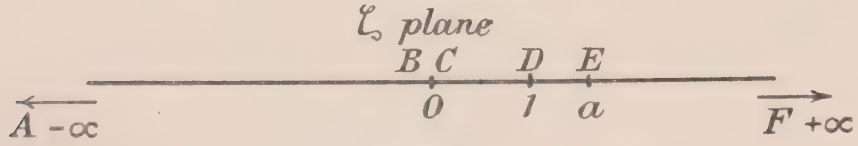


FIG. 127.

values of K and a may be determined. Writing $\zeta = re^{j\theta}$, assume r to remain constant.

$$\frac{d\zeta}{d\theta} = jre^{j\theta} = j\zeta$$

In moving from a point slightly to the right of the origin in Fig. 127 to a point slightly to the left, θ changes by π. At the same time the corresponding point in the z plane changes by jb. Hence $\frac{dz}{d\theta} = \frac{jb}{\pi}$. But $\frac{dz}{d\zeta} = \frac{dz}{d\theta} \div \frac{d\zeta}{d\theta} = \frac{b}{\pi\zeta}$.

When ζ is very small equation 831 gives $\frac{dz}{d\zeta} = \frac{K}{\zeta\sqrt{a}}$. Hence

$K = \frac{b\sqrt{a}}{\pi}$ or

$$a = \frac{\pi^2 K^2}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (832)$$

Again when ζ is very great $\frac{dz}{d\zeta} = \frac{K}{\zeta}$. In moving similarly from a point far to the right of the origin in Fig. 127 to a similarly distant point on the left θ again changes by π and the corresponding change in z is jc.

Hence $\frac{dz}{d\theta} = \frac{jc}{\pi}$. Hence $\frac{dz}{d\zeta} = \frac{c}{\pi\zeta} = \frac{K}{\zeta}$ or

$$K = \frac{c}{\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (833)$$

and

$$a = \frac{c^2}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (834)$$

Hence equation 831 becomes

$$\frac{dz}{d\zeta} = \frac{c}{\pi\zeta}\sqrt{\frac{\zeta - 1}{\zeta - a}} \quad . \quad . \quad . \quad . \quad . \quad (835)$$

or

$$z = \frac{c}{\pi}\int\frac{1}{\zeta}\sqrt{\frac{\zeta - 1}{\zeta - a}}d\zeta \quad . \quad . \quad . \quad . \quad (836)$$

Two methods of integrating this will be given.

(*a*) Write $\sqrt{\frac{\zeta-1}{\zeta-a}} = \frac{1}{\theta}$ giving $\zeta = \frac{\theta^2-a}{\theta^2-1}$

then $$d\zeta = \frac{2\theta(a-1)}{(\theta^2-1)^2}\,d\theta$$

and the integral becomes

$$z = 2K\int \frac{a-1}{(\theta^2-a)(\theta^2-1)}\,d\theta$$

$$= 2K\int\left(\frac{1}{1-\theta^2} - \frac{1}{a-\theta^2}\right)d\theta$$

$$\text{or } z = 2K\left(\coth^{-1}\theta - \frac{1}{\sqrt{a}}\coth^{-1}\frac{\theta}{\sqrt{a}}\right) + c$$

$$\text{or } z = 2K\left[\coth^{-1}\sqrt{\frac{\zeta-a}{\zeta-1}} - \frac{1}{\sqrt{a}}\coth^{-1}\sqrt{\frac{\zeta-a}{a(\zeta-1)}}\right] + \mathrm{C}$$

When $\zeta = 1$, $z = 0$ and therefore $\mathrm{C} = 0$. Unfortunately this form is not particularly suitable for computation. It may easily be transformed. Thus

$$2\coth^{-1}\sqrt{\frac{p}{q}} = \log\frac{\sqrt{\frac{p}{q}}+1}{\sqrt{\frac{p}{q}}-1} = \log\frac{q+p+2\sqrt{pq}}{p-q}$$

$$= \log\frac{q+p+\sqrt{(q+p)^2-(q-p)^2}}{q-p} - \log(-1) = \cosh\frac{q+p}{q-p} - j\pi$$

So that

$$z = K\left[\cosh^{-1}\frac{2\zeta-(a+1)}{a-1}\right.$$

$$\left. - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{(1+a)\zeta-2a}{(a-1)\zeta} - j\pi\left(1-\frac{1}{\sqrt{a}}\right)\right]$$

The same result may be obtained directly as follows:

(*b*) $$z = \frac{c}{\pi}\int\frac{1}{\zeta}\sqrt{\frac{\zeta-1}{\zeta-a}}\,d\zeta$$

$$= \frac{c}{\pi}\left[\int\frac{d\zeta}{\sqrt{(\zeta-a)(\zeta-1)}} - \int\frac{d\zeta}{\zeta\sqrt{(\zeta-a)(\zeta-1)}}\right]$$

$$\int\frac{d\zeta}{\sqrt{(\zeta-a)(\zeta-1)}} = \cosh^{-1}\frac{2\zeta-(a+1)}{a-1}$$

To determine $\int \frac{d\zeta}{\zeta\sqrt{(\zeta-a)(\zeta-1)}}$ write $\zeta = \frac{1}{w}$, then

$$d\zeta = -\frac{1}{w^2}\,dw$$

and the integral becomes

$$\int \frac{dw}{\sqrt{(1-aw)(1-w)}} = \frac{1}{\sqrt{a}} \cosh^{-1} \frac{2w - \frac{1}{a} - 1}{\frac{1}{a} - 1}$$

Substituting back gives

$$\int \frac{d\zeta}{\zeta\sqrt{(\zeta-a)(\zeta-1)}} = \frac{1}{\sqrt{a}} \cosh^{-1} \frac{(1+a)\zeta - 2a}{(a-1)\zeta}$$

Hence

$$z = \frac{c}{\pi}\left[\cosh^{-1}\frac{2\zeta-(a+1)}{a-1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{(1+a)\zeta-2a}{(a-1)\zeta}\right] + C \quad . \quad (837)$$

The constant of integration may be complex. It is given by the condition that $\zeta = 1$ when $z = 0$. Substituting these values gives

$$C = -\frac{c}{\pi}\left(1 - \frac{1}{\sqrt{a}}\right)\cosh^{-1}(-1)$$

But $\cosh^{-1}(-1) = j\pi$ (p. 11) whence

$$C = -jc\left(1 - \frac{1}{\sqrt{a}}\right)$$

To eliminate one constant from further work it is assumed that $\rho = 1$; then the current function becomes identical with the stream function. There is no real loss of generality thereby incurred since in the final result potentials may be multiplied by ρ. In transforming into the W plane it is assumed that $U = 0$ along $CDEF$ and $U = I$, the total current carried by the conductor, along BA. Writing $W = V + jU$ and also $W = \lambda \log \zeta$, then the difference in current between AB and $CDEF$ is, by equation 829, $I = \lambda\pi$ or $\lambda = \frac{I}{\pi}$. Then

$$W = V + jU = \frac{I}{\pi}\log\zeta \quad . \quad . \quad . \quad . \quad (838)$$

No constant of integration has been included; when $\zeta = 1$, $W = 0$. The origin in the W plane therefore corresponds to

the corner D and the diagram in the W plane is Fig. 128. The effect of the transformation from the ζ plane (Fig. 127) to the W plane (Fig. 128) has been to lift BA and turn it through

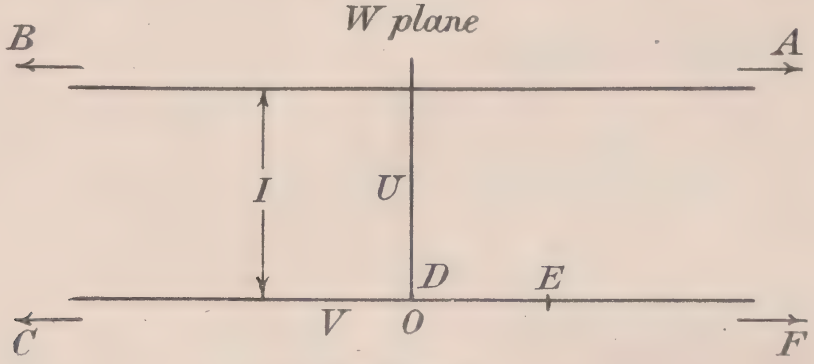


FIG. 128.

an angle π so that it is placed above the axis $U = 0$ by a distance I. Equation 838 gives $\zeta = e^{\pi W/I}$. Making this substitution gives finally

$$z = \frac{c}{\pi}\left[\cosh^{-1}\frac{2e^{\pi W/I}-(a+1)}{a-1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{(a+1)-2ae^{-\pi W/I}}{a-1} - j\pi\left(1-\frac{1}{\sqrt{a}}\right)\right] \quad . \quad (839)$$

Practical Example.

To determine the current and potential distribution when $I = 1$ ampere, $b = 5$ cm., $c = 10$ cm., $\rho = 1$.

Then $a = 4$, $\sqrt{a} = 2$.

$e^{\pi W/I} = e^{\pi W} = e^{\pi(V+jU)} = e^{\pi V}(\cos \pi U + j \sin \pi U)$

$e^{-\pi W/I} = e^{-\pi V}(\cos \pi U - j \sin \pi U)$ and equation 839 gives

$$z = \frac{10}{\pi}\left[\cosh^{-1}\left\{\frac{2}{3}e^{\pi V}(\cos \pi U + j \sin \pi U) - \frac{5}{3}\right\} - \frac{1}{2}\cosh^{-1}\left\{\frac{5}{3} - \frac{8}{3}e^{-\pi V}(\cos \pi U - j \sin \pi U)\right\} - j\frac{\pi}{2}\right] \quad . \quad (840)$$

Tables of complex hyperbolic functions are of considerable assistance in evaluating this expression. It may however be evaluated directly. Thus, considering the equipotential $V = 0$, the point where $U = 0{\cdot}5$ is given by

$$z = \frac{10}{\pi}\left[\cosh^{-1}\left\{\frac{2}{3}(0+j) - \frac{5}{3}\right\} - \frac{1}{2}\cosh^{-1}\left\{\frac{5}{3} - \frac{8}{3}(0-j)\right\} - j\frac{\pi}{2}\right]$$

or $$z = \frac{10}{\pi}\left[\cosh^{-1}\left(-\frac{5}{3}+j\frac{2}{3}\right) - \frac{1}{2}\cosh^{-1}\left(\frac{5}{3}+j\frac{8}{3}\right) - j\frac{\pi}{2}\right]$$

Using the relation

$$\cosh^{-1}(u+jv) = \cosh^{-1}\left\{\frac{\sqrt{(1+u)^2+v^2}+\sqrt{(1-u)^2+v^2}}{2}\right\}$$

$$+ j\cos^{-1}\left\{\frac{\sqrt{(1+u)^2+v^2}-\sqrt{(1-u)^2+v^2}}{2}\right\} \quad . \quad . \quad (841)$$

one obtains

$$\cosh^{-1}\left(-\frac{5}{3}+j\frac{2}{3}\right) = \cosh^{-1} 1{\cdot}8458 + j\cos^{-1}(-0{\cdot}9030)$$

$$= 1{\cdot}223 + j\,2{\cdot}698$$

$$\cosh^{-1}\left(\frac{5}{3}+j\frac{8}{3}\right) = \cosh^{-1} 3{\cdot}2600 + j\cos^{-1}(0{\cdot}5113)$$

$$= 1{\cdot}850_5 + j\,1{\cdot}034$$

giving $$z = \frac{10}{\pi}(0{\cdot}298 + j\,0{\cdot}610)$$

or $$z = x + jy = 0{\cdot}949 + j\,1{\cdot}942$$

Other points on this equipotential may be similarly calculated. The point on the same stream line $U = 0{\cdot}5$ where $V = 1$ will be determined. Then $e^{\pi V} = 23{\cdot}141$ and $e^{-\pi V} = 0{\cdot}04321$

$$z = \frac{10}{\pi}\left[\cosh^{-1}(-1{\cdot}667 + j15{\cdot}427)\right.$$

$$\left. - \frac{1}{2}\cosh^{-1}(1{\cdot}667 + j\,0{\cdot}1152) - j\frac{\pi}{2}\right]$$

giving $$z = x + jy = 9{\cdot}177 + j\,0{\cdot}131$$

Obtaining several points in this way the stream lines and

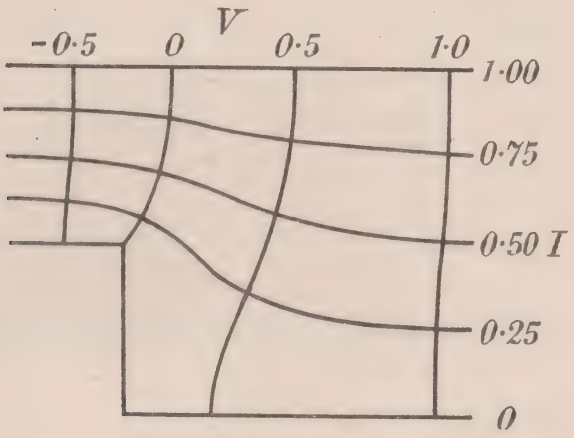


FIG. 129.

equipotentials can be plotted. They are shown in Fig. 129. At a distance of a few breadths away from the change of section

the current is practically uniformly distributed and the equipotentials become straight lines perpendicular to the sides of the strip.

Increase of Resistance due to Change of Section.

Owing to the non-uniform distribution of the current across the section the resistance is greater than the sum of the resistances of the separate parts, each considered as part of a conductor of infinite length. The correction can be determined as follows. Assume that 1 ampere is flowing along the conductor. Then along $CDEF$, $U = 0$ and $\frac{W}{I} = V$. Then equation 839 becomes

$$z = \frac{c}{\pi}\left[\cosh^{-1}\frac{2e^{\pi V} - (a+1)}{a-1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{(a+1) - 2ae^{-\pi V}}{a-1} - j\pi\left(1 - \frac{1}{\sqrt{a}}\right)\right] \quad . \quad . \quad (842)$$

Suppose V is positive and very big, say $= V_1$ then

$$z_1 \doteqdot \frac{c}{\pi}\left[\cosh^{-1}\frac{2}{a-1}e^{\pi V_1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{a+1}{a-1} - j\pi\left(1-\frac{1}{\sqrt{a}}\right)\right]$$

But if V_1 is very big one may write

$$\cosh\theta = \frac{e^\theta}{2} = \frac{2}{a-1}e^{\pi V_1}; \quad e^\theta = \frac{4}{a-1}e^{\pi V_1}$$

$$\theta = \log\frac{4}{a-1} + \pi V_1$$

whence

$$z_1 = \frac{c}{\pi}\left[\pi V_1 + \log\frac{4}{a-1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{a+1}{a-1} - j\pi\left(1-\frac{1}{\sqrt{a}}\right)\right]$$

or $$x_1 = \frac{c}{\pi}\left[\pi V_1 + \log\frac{4}{a-1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{a+1}{a-1}\right]$$

Suppose now that V is negative and very big, say $= -V_2$ then

$$z_2 \doteqdot \frac{c}{\pi}\left[\cosh^{-1}\left(-\frac{a+1}{a-1}\right) - \frac{1}{\sqrt{a}}\cosh^{-1}\left(-\frac{2a}{a-1}e^{\pi V_2}\right) - j\pi\left(1 - \frac{1}{\sqrt{a}}\right)\right]$$

Since $\cosh^{-1}(-\mu) = \cosh^{-1}\mu + j\pi$ (p. 11) it follows that

$$x_2 \doteqdot \frac{c}{\pi}\left[\cosh^{-1}\frac{a+1}{a-1} - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{2a}{a-1}e^{\pi V_2}\right]$$

or $$x_2 = \frac{c}{\sqrt{a}\pi}\left[-\pi V_2 + \sqrt{a}\cosh^{-1}\frac{a+1}{a-1} - \log\frac{4a}{a-1}\right]$$

If the current were uniformly distributed x_1 would be

$$x_1' = \frac{c}{\pi}(\pi V_1)$$

and x_2 would be

$$x_2' = \frac{c}{\sqrt{a}\pi}(-\pi V_2)$$

The effective length of the wider conductor has been increased by

$$\frac{c}{\pi}\left[\frac{1}{\sqrt{a}}\cosh^{-1}\frac{a+1}{a-1} - \log\frac{4}{a-1}\right]$$

and the effective length of the narrower conductor has been increased by

$$\frac{c}{\sqrt{a}\pi}\left[\sqrt{a}\cosh^{-1}\frac{a+1}{a-1} - \log\frac{4a}{a-1}\right]$$

but the narrower conductor has $\sqrt{a}$ times the resistance of the wider conductor. Expressing the whole increase in effective length in terms of the wider conductor it is

$$\lambda = \frac{c}{\pi}\left[\left(\frac{1}{\sqrt{a}} + \sqrt{a}\right)\cosh^{-1}\frac{a+1}{a-1} - \log\frac{16a}{(a-1)^2}\right] \quad . \quad (843)$$

This may be otherwise expressed

$$\frac{1}{\sqrt{a}} + \sqrt{a} = \frac{b^2+c^2}{bc}$$

$$\cosh^{-1}\frac{a+1}{a-1} = \log\left\{\frac{a+1}{a-1} + \sqrt{\left(\frac{a+1}{a-1}\right)^2 - 1}\right\}$$

$$= \log\frac{a+1+\sqrt{4a}}{a-1} = \log\frac{(c+b)^2}{c^2-b^2} = \log\frac{c+b}{c-b}$$

$$\log\frac{(a-1)^2}{16a} = 2\log\frac{a-1}{4\sqrt{a}} = 2\log\frac{c^2-b^2}{4cb}$$

Substituting these values in equation 843 gives

$$\lambda = \frac{c}{\pi}\left[2 \log \frac{c^2 - b^2}{4cb} + \frac{c^2 + b^2}{cb} \log \frac{c + b}{c - b}\right] \quad . \quad (844)$$

which is the expression given by Lees (*Phys. Soc. Proc.*, Vol. XXI, p. 739).

For a change to half width, i.e. $c = 2b$, $\lambda = 0{\cdot}25c$.

Flow of Current from a Point Source into a Sheet of Resistive Material.

If current flows from a point source into an extensive sheet of resistive material and is removed at the distant boundary, the lines of force and lines of flow are coincident radii, the equipotentials are circles. The transformation required to solve the problem by conjugate functions is suggested by equation 820.

The conditions on the plate are obtained directly from Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

by writing, from equation 813

$$V - jU = \phi(x + jy)$$

whence

$$\frac{\partial V}{\partial x} - j\frac{\partial U}{\partial x} = \phi'(x + jy)$$

$$\frac{\partial V}{\partial y} - j\frac{\partial U}{\partial y} = j\phi'(x + jy)$$

or

$$\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x} \text{ and } \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

Since

$$U = \rho I$$

$$\frac{\partial V}{\partial y} = \rho\frac{\partial I}{\partial x} \text{ and } \frac{\partial V}{\partial x} = -\rho\frac{\partial I}{\partial y}$$

agreeing with equations 817 and 818.

The particular transformation required for the problem is

$$W = V - jU = V - j\rho I = A \log (x + jy) \quad . \quad (845)$$

Writing

$$x + jy = re^{j\theta}$$

then

$$V - j\rho I = A \log r + jA\theta$$

$$V = A \log r$$

and equipotentials are given by r = constant. That is, they are circles.

$$I = -\frac{A\theta}{\rho}$$

and so stream lines are lines radiating from the origin. If the total current entering the plate is I then $A = -\frac{\rho I}{2\pi}$. The potential at any point is therefore $-\frac{\rho I}{2\pi}\log r$. The potential gradient is

$$-\frac{\partial V}{\partial r} = \frac{\rho I}{2\pi r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (846)$$

Flow of Current from a Point Source into a Sheet of Material having Different Conductivities along the Two Axes.

This problem arose in actual experimental work. A sheet of metal was known to have, under certain circumstances, an effective resistivity in one direction two or three times its effective resistivity at right angles. Measurements made under these circumstances gave results which were unexpected. Applying the analysis here described the experimental results were completely explained.

For such a material Laplace's equation no longer holds. V and U are no longer conjugate functions; a somewhat similar analysis is applicable and they may be termed quasi-conjugate functions. If ρ_1 is the resistivity along the axis of x and ρ_2 the resistivity along the axis of y then the equivalent equations to 817 and 818 are

$$\frac{\partial V}{\partial x} = -\rho_1\frac{\partial I}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad (847)$$

and

$$\frac{\partial V}{\partial y} = \rho_2\frac{\partial I}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (848)$$

It follows that if $\rho_1 \neq \rho_2$ that the stream lines are no longer coincident with the lines of force, except along the axes.

Differentiating equation 847 with respect to x and multiplying by ρ_2 gives

$$\rho_2\frac{\partial^2 V}{\partial x^2} = -\rho_1\rho_2\frac{\partial^2 I}{\partial x\partial y}$$

Differentiating equation 848 with respect to y and multiplying by ρ_1 gives

$$\rho_1 \frac{\partial^2 V}{\partial y^2} = \rho_1 \rho_2 \frac{\partial^2 I}{\partial y \, \partial x}$$

whence

$$\rho_2 \frac{\partial^2 V}{\partial x^2} + \rho_1 \frac{\partial^2 V}{\partial y^2} = 0 \quad . \quad . \quad . \quad . \quad (849)$$

If $\rho_1 = \rho_2$ this degenerates to Laplace's equation. A solution of equation 849 is

$$V = f\left(x + j\sqrt{\frac{\rho_2}{\rho_1}}y\right) + F\left(x - j\sqrt{\frac{\rho_2}{\rho_1}}y\right) \quad . \quad (850)$$

Following previous work write

$$V - jU = \phi\left(x + j\sqrt{\frac{\rho_2}{\rho_1}}y\right) \quad . \quad . \quad . \quad (851)$$

then $$\frac{\partial V}{\partial x} - j\frac{\partial U}{\partial x} = \phi'\left(x + j\sqrt{\frac{\rho_2}{\rho_1}}y\right)$$

and $$\frac{\partial V}{\partial y} - j\frac{\partial U}{\partial y} = j\sqrt{\frac{\rho_2}{\rho_1}}\phi'\left(x + j\sqrt{\frac{\rho_2}{\rho_1}}y\right)$$

giving $$j\sqrt{\frac{\rho_2}{\rho_1}}\frac{\partial V}{\partial x} + \sqrt{\frac{\rho_2}{\rho_1}}\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} - j\frac{\partial U}{\partial y}$$

$$\frac{\partial V}{\partial x} = -\sqrt{\frac{\rho_1}{\rho_2}}\frac{\partial U}{\partial y} \quad . \quad . \quad . \quad . \quad . \quad (852)$$

$$\frac{\partial V}{\partial y} = \sqrt{\frac{\rho_2}{\rho_1}}\frac{\partial U}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (853)$$

These equations must be identical with equations 847 and 848 and so U must be interpreted as $I\sqrt{\rho_1\rho_2}$. Then equation 851 becomes

$$V - j\sqrt{\rho_1\rho_2}I = \phi\left(x + j\sqrt{\frac{\rho_2}{\rho_1}}y\right) \quad . \quad . \quad . \quad (854)$$

Making the logarithmic transformation gives

$$V - j\sqrt{\rho_1\rho_2}I = A \log\left(x + j\sqrt{\frac{\rho_2}{\rho_1}}y\right) \quad . \quad . \quad (855)$$

This compares with equation 845 for the isotropic sheet.

Write $$x + j\sqrt{\frac{\rho_2}{\rho_1}}y = a(\cos\alpha + j\sin\alpha) = ae^{j\alpha}.$$

a and α are connected with the polar co-ordinates r and θ as indicated in Fig. 130 (ρ_2 is assumed to be greater than ρ_1)

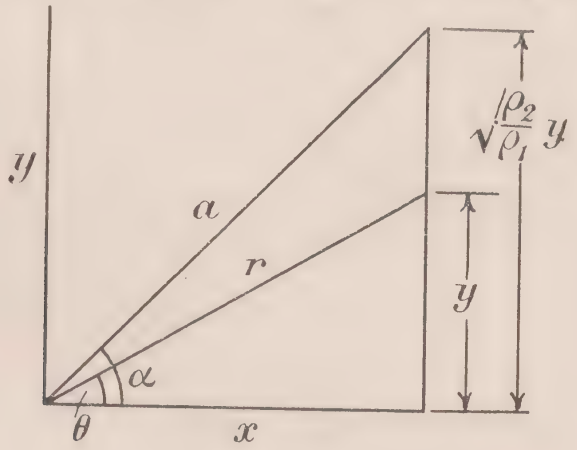


FIG. 130.

$$a = \sqrt{x^2 + \frac{\rho_2}{\rho_1}y^2}\,; \quad \tan\alpha = \frac{y}{x}\sqrt{\frac{\rho_2}{\rho_1}}$$

Then
$$V - j\sqrt{\rho_1\rho_2}I = A\log a + jA\alpha$$

$$V = A\log a\,; \quad I = -\frac{A\alpha}{\sqrt{\rho_1\rho_2}}$$

The equipotentials are given by $\log a$ constant or a constant

$$x^2 + \frac{\rho_2}{\rho_1}y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad (856)$$

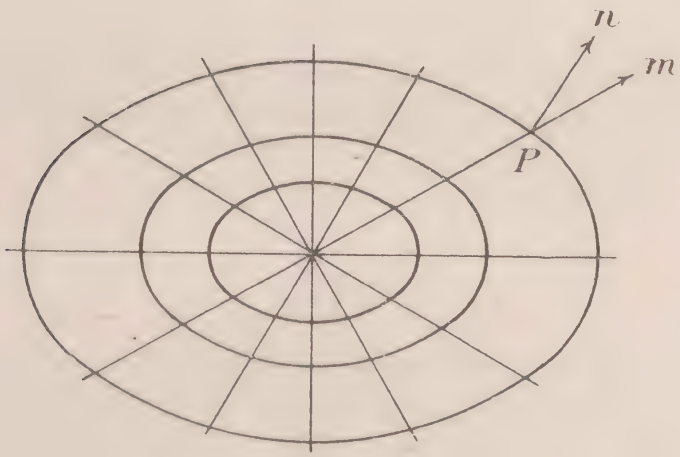


FIG. 131.

If a is constant, equation 856 is an ellipse of semi-major axis a and of semi-minor axis $a\sqrt{\frac{\rho_1}{\rho_2}}$. All equipotentials have the same ratio of axes. Stream lines are given by α constant and these are lines radiating from the origin. Stream lines and equipotentials are shown in Fig. 131. The lines of force

are of course normal to the equipotentials, but the lines of flow are no longer normal to the equipotentials. This is readily understood. It has been assumed that $\rho_2 > \rho_1$, that is the material conducts better parallel to the axis of x than it does parallel to the axis of y. At P (Fig. 131) the electric force is along n. Because of the better conductivity in the x direction the resulting current flows in the direction m.

If the total current flowing in at the source is I then

$$A = -\frac{\sqrt{\rho_1\rho_2}I}{2\pi}$$ and the potential at any point is

$$V = -\frac{\sqrt{\rho_1\rho_2}I}{2\pi}\log a = -\frac{\sqrt{\rho_1\rho_2}I}{2\pi}\log (r\cos\theta\sec\alpha)$$

$$= -\frac{\sqrt{\rho_1\rho_2}I}{2\pi}(\log r + \log\cos\theta\sec\alpha)$$

The potential gradient along the radius vector is

$$-\frac{\partial V}{\partial r} = \frac{\sqrt{\rho_1\rho_2}I}{2\pi r} \quad . \quad . \quad . \quad . \quad . \quad (857)$$

This is the same as in a uniform plate for which $\rho = \sqrt{\rho_1\rho_2}$.

At a given distance from the source *the potential gradient along the radius vector is independent of the orientation.*

This leads to a somewhat unexpected, but experimentally confirmed, result. If four electrodes $APQB$ are placed in line on an extensive plate having different conductivities in two different directions (Fig. 132) and a given current is fed in at

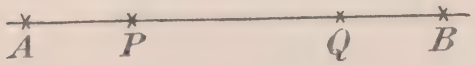


FIG. 132.

A and out at B the potential difference between P and Q is independent of the orientation. If the direction AB is the highly resistive direction then the current spreads out very much in the direction at right angles to AB and the current density along PQ is small. The difference of potential between P and Q is due to a low current density through material of high resistivity. If $APQB$ be turned through a right angle the same total current gives a higher current density between P and Q. This, with the lowered resistivity, gives the same difference of potential.

APPENDIX I

Circular, Hyperbolic and Exponential Functions

$$\cos^2\alpha + \sin^2\alpha = 1$$
$$\sec^2\alpha = 1 + \tan^2\alpha$$
$$\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$$
$$\sin(-\alpha) = -\sin\alpha$$
$$\cos(-\alpha) = \cos\alpha$$

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right); \quad \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$$

$$\sin\alpha = -\cos\left(\alpha + \frac{\pi}{2}\right)$$

$$\cos\alpha = \sin\left(\alpha + \frac{\pi}{2}\right)$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\beta - \alpha}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\beta - \alpha}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha + \beta}{2}\sin\frac{\beta - \alpha}{2}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$
$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$
$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}; \quad \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$
$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \ldots \qquad (x < 1)$$

$$e = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} \ldots = 2{\cdot}71828 \ldots$$

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots$$

$$a^x = 1 + x \log a + \frac{x^2}{\lfloor 2} (\log a)^2 + \frac{x^3}{\lfloor 3} (\log a)^3 + \ldots$$

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad (x^2 < 1)$$

$$\log x = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots \qquad (2 > x > 0)$$

$$\cos \theta + j \sin \theta = e^{j\theta}; \quad \cos \theta - j \sin \theta = \varepsilon^{-j\theta}$$
$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \ldots$$
$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \ldots$$

$$\cosh \phi = \frac{e^{\phi} + e^{-\phi}}{2} = 1 + \frac{\phi^2}{\lfloor 2} + \frac{\phi^4}{\lfloor 4} + \frac{\phi^6}{\lfloor 6} \ldots = \cosh (-\phi)$$
$$\sinh \phi = \frac{e^{\phi} - e^{-\phi}}{2} = \phi + \frac{\phi^3}{\lfloor 3} + \frac{\phi^5}{\lfloor 5} + \frac{\phi^7}{\lfloor 7} \ldots = -\sinh (-\phi)$$

$$\cosh \phi + \sinh \phi = e^{\phi}$$
$$\cosh^2 \phi - \sinh^2 \phi = 1$$
$$\operatorname{sech}^2 \phi = 1 - \tanh^2 \phi$$
$$\operatorname{cosech}^2 \phi = \coth^2 \phi - 1$$

$$\tanh \phi = \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}} = \phi - \frac{1}{3}\phi^3 + \frac{2}{15}\phi^5 - \frac{17}{315}\phi^7 + \ldots \left(\phi^2 < \frac{\pi^2}{4}\right)$$

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$$
$$\sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$$
$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$$
$$\cosh(\alpha-\beta) = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha+\tanh\beta}{1+\tanh\alpha\tanh\beta}$$
$$\tanh(\alpha-\beta) = \frac{\tanh\alpha-\tanh\beta}{1-\tanh\alpha\tanh\beta}$$

$$\sinh\alpha\sinh\beta = \frac{1}{2}[\cosh(\alpha+\beta) - \cosh(\alpha-\beta)]$$
$$\cosh\alpha\cosh\beta = \frac{1}{2}[\cosh(\alpha+\beta) + \cosh(\alpha-\beta)]$$
$$\sinh\alpha\cosh\beta = \frac{1}{2}[\sinh(\alpha+\beta) + \sinh(\alpha-\beta)]$$

$$\sinh\alpha + \sinh\beta = 2\sinh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2}$$
$$\cosh\alpha + \cosh\beta = 2\cosh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2}$$
$$\cosh\alpha - \cosh\beta = 2\sinh\frac{\alpha+\beta}{2}\sinh\frac{\alpha-\beta}{2}$$

$$\cosh 2\alpha = \cosh^2\alpha + \sinh^2\alpha$$
$$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha$$

$$\sinh 3\alpha = 3\sinh\alpha + 4\sinh^3\alpha$$
$$\cosh 3\alpha = 4\cosh^3\alpha - 3\cosh\alpha$$

$$\cosh j\alpha = \cos\alpha\,; \quad \cos j\alpha = \cosh\alpha$$
$$\sinh j\alpha = j\sin\alpha\,; \quad \sin j\alpha = j\sinh\alpha$$

$$\sinh(\alpha+j\beta) = \sinh\alpha\cos\beta + j\cosh\alpha\sin\beta$$
$$\sinh(\alpha-j\beta) = \sinh\alpha\cos\beta - j\cosh\alpha\sin\beta$$
$$\cosh(\alpha+j\beta) = \cosh\alpha\cos\beta + j\sinh\alpha\sin\beta$$
$$\cosh(\alpha-j\beta) = \cosh\alpha\cos\beta - j\sinh\alpha\sin\beta$$

$$\sinh^{-1} \quad x = \log (x + \sqrt{x^2 + 1})$$

$$\cosh^{-1} \quad x = \log (x + \sqrt{x^2 - 1})$$

$$\tanh^{-1} \quad x = \frac{1}{2} \log \frac{1 + x}{1 - x}$$

$$\coth^{-1} \quad x = \frac{1}{2} \log \frac{x + 1}{x - 1}$$

$$\operatorname{sech}^{-1} \quad x = \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right)$$

$$\operatorname{cosech}^{-1} x = \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right)$$

$$\sinh^{-1} (x \pm jy) = \cosh^{-1} \left[\frac{\sqrt{(1 + y)^2 + x^2} + \sqrt{(1 - y)^2 + x^2}}{2} \right]$$

$$\pm j \sin^{-1} \left[\frac{\sqrt{(1 + y)^2 + x^2} - \sqrt{(1 - y)^2 + x^2}}{2} \right]$$

$$\cosh^{-1} (x \pm jy) = \cosh^{-1} \left[\frac{\sqrt{(1 + x)^2 + y^2} + \sqrt{(1 - x)^2 + y^2}}{2} \right]$$

$$\pm j \cos^{-1} \left[\frac{\sqrt{(1 + x)^2 + y^2} - \sqrt{(1 - x)^2 + y^2}}{2} \right]$$

$$\tanh^{-1} (x \pm jy) = \frac{1}{2} \log \sqrt{\frac{(1 + x)^2 + y^2}{(1 - x)^2 + y^2}}$$

$$\pm \frac{j}{2} \left[\pi \mp \tan^{-1} \left(\frac{1 + x}{y} \right) \mp \tan^{-1} \left(\frac{1 - x}{y} \right) \right]$$

$$\cosh^{-1} (-x) = \cosh^{-1} x + j\pi$$

$$\log (-x) \quad = \log x + j\pi$$

Differentiation.

If $y = x^n$ $\qquad \dfrac{dy}{dx} = nx^{n-1}$

$y = e^x$ $\qquad \dfrac{dy}{dx} = e^x$

$y = a^x$ $\qquad \dfrac{dy}{dx} = a^x \log a$

$y = \log x$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x}$

$y = \sin x$ $\qquad \dfrac{dy}{dx} = \cos x = \sin \left(x + \dfrac{\pi}{2} \right)$

$y = \cos x$ $\qquad \dfrac{dy}{dx} = -\sin x = \cos \left(x + \dfrac{\pi}{2} \right)$

If $y = \tan x$	$\frac{dy}{dx} = \sec^2 x$
$y = \cot x$	$\frac{dy}{dx} = -\operatorname{cosec}^2 x$
$y = \sec x$	$\frac{dy}{dx} = \sec x \,.\, \tan x$
$y = \operatorname{cosec} x$	$\frac{dy}{dx} = -\operatorname{cosec} x \,.\, \cot x$
$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$
$y = \sin^{-1} \frac{x}{a}$	$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$
$y = \cos^{-1} \frac{x}{a}$	$\frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$
$y = \tan^{-1} \frac{x}{a}$	$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$
$y = \cot^{-1} \frac{x}{a}$	$\frac{dy}{dx} = -\frac{a}{a^2 + x^2}$
$y = \sec^{-1} \frac{x}{a}$	$\frac{dy}{dx} = \frac{a}{x\sqrt{x^2 - a^2}}$
$y = \operatorname{cosec}^{-1} \frac{x}{a}$	$\frac{dy}{dx} = -\frac{a}{x\sqrt{x^2 - a^2}}$
$y = \sinh x$	$\frac{dy}{dx} = \cosh x$
$y = \cosh x$	$\frac{dy}{dx} = \sinh x$
$y = \tanh x$	$\frac{dy}{dx} = \operatorname{sech}^2 x$
$y = \coth x$	$\frac{dy}{dx} = -\operatorname{cosech}^2 x$
$y = \operatorname{sech} x$	$\frac{dy}{dx} = -\operatorname{sech} x \tanh x$
$y = \operatorname{cosech} x$	$\frac{dy}{dx} = -\operatorname{cosech} x \coth x$
$y = \sinh^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$
$y = \cosh^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$

If $y = \tanh^{-1} x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$

$y = \coth^{-1} x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$

$y = \operatorname{sech}^{-1} x$ $\quad \dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{1 - x^2}}$

$y = \operatorname{cosech}^{-1} x$ $\quad \dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2 + 1}}$

$y = e^{ax} \sin(\omega x + \phi)$ $\quad \dfrac{dy}{dx} = re^{ax} \sin(\omega x + \phi + \theta)$

where $r = \sqrt{\omega^2 + a^2}$ $\quad \theta = \tan^{-1} \omega/a$

If $y = uv$ $\quad \dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

$y = \dfrac{u}{v}$ $\quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

If $y = uv$ then

$$y_n = u_n v + nu_{n-1}v_1 + \frac{n(n-1)}{\lfloor 2}u_{n-2}v_2 \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}u_{n-r}v_r \ldots \text{ (Leibnitz)}$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \ldots \text{ (Maclaurin)}$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \ldots \text{ (Taylor)}$$

Integration (integration constants omitted).

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{dx}{x} = \log x$$

$$\int e^x\,dx = e^x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \tan x\,dx = \log(\sec x)$$

$$\int \cot x\,dx = \log(\sin x)$$

$$\int \sec x\,dx = \log(\sec x + \tan x)$$

$$\int \operatorname{cosec} x\,dx = \log\left(\tan\frac{x}{2}\right)$$

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2}$$

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2}$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \tanh x\,dx = \log\cosh x$$

$$\int \coth x\,dx = \log\sinh x$$

$$\int \operatorname{sech} x\,dx = 2\tan^{-1} \varepsilon^{x}$$

$$\int \operatorname{cosech} x\,dx = \log\tanh\frac{x}{2}$$

$$\int \sinh^{-1} x\,dx = x\sinh^{-1} x - \sqrt{1+x^2}$$

$$\int \cosh^{-1} x\,dx = x\cosh^{-1} x - \sqrt{x^2-1}$$

$$\int \tanh^{-1} x\,dx = x\tanh^{-1} x + \log\sqrt{1-x^2}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} = \log\frac{x+\sqrt{x^2-a^2}}{a}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} = \log\frac{x+\sqrt{x^2+a^2}}{a}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} = \frac{1}{2a} \log \frac{a + x}{a - x}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int e^{ax} \sin (\omega x + \phi) dx = \frac{1}{r} e^{ax} \sin (\omega x + \phi - \theta)$$

where $r = \sqrt{\omega^2 + a^2}$ $\qquad$ $\theta = \tan^{-1} \omega/a$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Beta Functions.

$$B(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1} dx$$

$$B(m, n) = B(n, m)$$

$$B(m, n) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta$$

Gamma Functions.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n) = (n - 1)\Gamma(n - 1)$$

If n is a positive integer $\Gamma(n) = \underline{|n - 1}$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Integral Values of the Gamma Functions

$$\Gamma(p) = \underline{|p - 1}$$
$$\Gamma(1) = 1$$
$$\Gamma(0) = \infty$$
$$\Gamma(-p) = \infty$$

where p is a positive integer.

Relation between the B and Γ Functions.

$$B(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m + n)}$$

Particular Integrals.

$$\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{n+2}{2}\right)$$

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Operational Forms.

$$D \equiv \frac{d}{dx},\ D^{-1} \equiv \int(\quad)dx + c$$

$$(D-a)(D-b) \equiv (D-b)(D-a)$$

$$F(D) \equiv a_0 + a_1 D + a_2 D^2 \ \ldots\ a_n D^n \equiv \sum_0^n a_k D^k$$

$$F(D)\,.\,f(D) = F(D) \times f(D) = f(D)\,.\,F(D)$$

$$(D-a)^{-1} \equiv e^{ax} D^{-1} e^{-ax}$$

When $D \equiv \dfrac{\partial}{\partial x}$ and $p \equiv \dfrac{\partial}{\partial t}$

$$D + p \equiv p + D$$

$$D\,.\,p \equiv p\,.\,D$$

$$F(D)e^{\phi(x)} \equiv e^{\phi(x)}F[D + \phi'(x)]$$

$$F^{-1}(D)e^{\phi(x)} \equiv e^{\phi(x)}F^{-1}[D + \phi'(x)]$$

$$F(D)e^{ax} \equiv e^{ax}F(D + a)$$

$$F^{-1}(D)e^{ax} \equiv e^{ax}F^{-1}(D + a)$$

Operations upon Unity.

$$F(D)e^{ax}\,.\,1 = e^{ax}F(a)$$

$$F^{-1}(D)e^{ax}\,.\,1 = e^{ax}F^{-1}(a)$$

$$F^{-1}(D + a)\,.\,1 = F^{-1}(a)$$

If $F(D) = (D-a)^r\phi(D)$ then

$$F^{-1}(D)e^{ax}\,.\,1 = \frac{1}{\phi(a)}\frac{x^r e^{ax}}{\lfloor r}$$

Circular and Hyperbolic Functions.

$$F(D^2)\begin{matrix}\sin ax \\ \cos ax\end{matrix} = F(-a^2)\begin{matrix}\sin ax \\ \cos ax\end{matrix}$$

$$F(D^2)\begin{matrix}\sinh ax \\ \cosh ax\end{matrix} = F(a^2)\begin{matrix}\sinh ax \\ \cosh ax\end{matrix}$$

$$\frac{1}{D^2 + a^2}\sin ax = -\frac{x}{2a}\cos ax$$

$$\frac{1}{D^2 + a^2}\cos ax = \frac{x}{2a}\sin ax$$

The Linear Differential Equation.

$F(D) \,.\, y = X$

Solution $y = F^{-1}(D) \,.\, X + F^{-1}(D) \,.\, 0$

$F^{-1}(D) \,.\, X$ is particular integral

$F^{-1}(D) \,.\, 0$ is complementary function

Operations upon Zero.

$$D^{-n} \,.\, 0 = \sum_{0}^{n-1} B_r x^r$$

$$(D - m)^{-k} \,.\, 0 = e^{mx} \sum_{0}^{k-1} B_r x^r$$

$$(D^2 + a^2)^{-1} \,.\, 0 = A \cos ax + B \sin ax$$

$$(D^2 + aD + b)^{-1} \,.\, 0$$

$$= Ae^{\frac{-a+\sqrt{a^2-4b}}{2}x} + Be^{\frac{-a-\sqrt{a^2-4b}}{2}x} \text{ when } a^2 > 4b$$

$$\text{or } e^{-\frac{ax}{2}}\left[A \cos \sqrt{b - \frac{a^2}{4}}\, x + B \sin \sqrt{b - \frac{a^2}{4}}\, x\right] \text{ when } a^2 < 4b$$

$$\text{or } e^{-\frac{ax}{2}}(A + Bx) \text{ when } a^2 = 4b.$$

Inverse Operations upon Unity.

$F(D) \,.\, 1 = F(0)$

$F^{-1}(D) \,.\, 1 = F^{-1}(0)$

If $F(D) = D^n \phi(D)$

$$F^{-1}(D) \,.\, 1 = \frac{x^n}{\phi(0) \lfloor n}$$

Bessel Functions.

Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

Solutions $J_n(x)$ and $Y_n(x)$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(\lfloor 2)^2} - \frac{x^6}{2^6(\lfloor 3)^2} + \cdots$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{2r}}{(\lfloor r)^2}$$

$$Y_0(x) = \frac{2}{\pi}\{\log (x/2) + \gamma\} J_0(x) - \frac{2}{\pi}\sum_{r=1}^{\infty}(-1)^r\left(1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{r}\right)$$

$$\gamma = \underset{m \to \infty}{\text{Limit}}\left[1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{m} - \log m\right] \doteqdot 0{\cdot}5772$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{(x/2)^2}{\Gamma(n+2)} + \frac{(x/2)^4}{\lfloor 2\Gamma(n+3)} \cdots\right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{\lfloor r\Gamma(n+r+1)}$$

When n is integral

$$Y_n(x) = \frac{2}{\pi}\{\gamma + \log(x/2)\}J_n(x) - \frac{1}{\pi}\sum_{r=}^{n-1} \frac{\lfloor n-r-1}{\lfloor r}\left(\frac{2}{x}\right)^{n-2r}$$

$$- \frac{1}{\pi}\sum_{r=0}^{\infty}(-1)^r \frac{(x/2)^{2r+n}}{\lfloor r \lfloor n+r}\left\{1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{r}+1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{n+r}\right\}$$

When n is not integral

$$Y_n = \frac{\cos n\pi J_n - J_{-n}}{\sin n\pi}$$

The Modified Bessel Functions.

These are solutions of

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right)y = 0$$

and are represented as $I_n(x)$ and $K_n(x)$

$$I_0(x) = 1 + \left(\frac{x}{2}\right)^2 + \frac{(x/2)^4}{(\lfloor 2)^2} + \frac{(x/2)^6}{(\lfloor 3)^2} + \cdots$$

$$I_n(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{n+2r}}{\lfloor r\Gamma(n+r+1)}$$

$$K_0(x) = -\left(\gamma + \log\frac{x}{2}\right)I_0(x) + \sum_{r=1}^{\infty}\frac{(x/2)^{2r}}{(\lfloor r)^2}\left\{1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{r}\right\}$$

When n is integral

$$K_n(x) = \frac{1}{2}\sum_{r=0}^{n-1}(-1)^r \frac{\lfloor n-r-1}{\lfloor r}\left(\frac{2}{x}\right)^{n-2r}$$

$$+ (-1)^{n+1}\sum_{r=0}^{\infty}\frac{(x/2)^{n+2r}}{\lfloor r \lfloor n+r}\left\{\log\frac{x}{2} - \frac{1}{2}[\psi(r+1)+\psi(n+r+1)]\right\}$$

where $\psi(r+1) = \left(1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{r}\right) - \gamma$

When n is not integral

$$K_n(x) = \frac{\frac{\pi}{2}\{I_{-n}(x) - I_n(x)\}}{\sin n\pi}$$

Integral Forms of the Bessel Functions.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$$

For a list of integral forms of $J_0(x)$ and $I_0(x)$ see p. 242.

Ber, Bei, Ker and Kei Functions.

See p. 246 *et seq.*

Fourier Series.

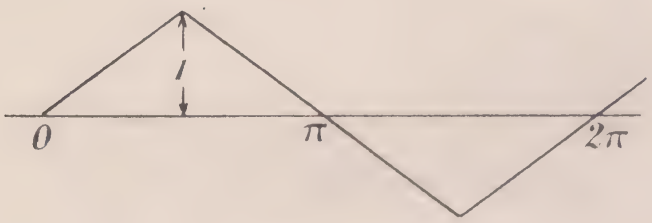


FIG. 133.

$$y = \frac{8}{\pi^2}\left(\sin\theta - \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta - \ldots\right)$$

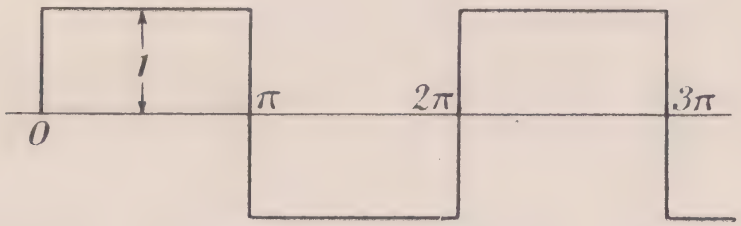


FIG. 134.

$$y = \frac{4}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \ldots\right)$$

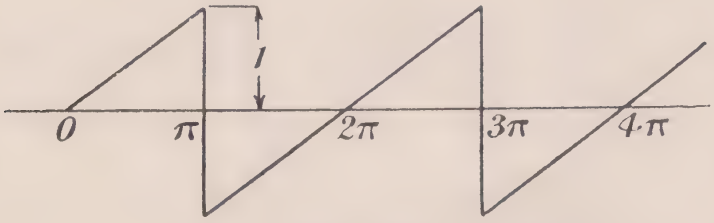


FIG. 135.

$$y = \frac{2}{\pi}\left(\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \frac{1}{4}\sin 4\theta + \ldots\right)$$

Equivalent Operators.

$$\frac{1}{p+a}\,1 = \frac{1-e^{-at}}{a}\,1$$

$$\frac{p}{p+a}\,1 = e^{-at}\,1$$

$$\frac{1}{p^n}\,1 = \frac{t^n}{\lfloor n}\,1$$

Expansion Theorem.

$$\frac{f_1(p)}{f_2(p)}\,1 = \left[\frac{f_1(0)}{f_2(0)} + \sum_{a,b,c,\ldots} \frac{f_1(p)e^{pt}}{pf_2'(p)}\right] 1$$

Duhamel's Theorem.

$$f[E(t)] = E(0)U(t) + \int_0^t E'(t') \,.\, U(t-t')dt'$$

$$= E(t)U(0) + \int_0^t E(t') \,.\, U'(t-t')dt'$$

APPENDIX II

Electrical and Magnetic Equations.

Force between charges $F = \frac{q_1 q_2}{\kappa r^2}$

Poisson's equation

$$\text{div. } \varepsilon = 4\pi\rho$$

or $\nabla^2 V + 4\pi\rho = 0$

Laplace's equation

$\nabla^2 V = 0$

Surface density of charge

$$\sigma = \frac{\kappa \varepsilon_s}{4\pi} = -\frac{\kappa}{4\pi}\frac{\partial V}{\partial n}$$

Displacement or electric flux density

$$D = \frac{\kappa\varepsilon}{4\pi}$$

Capacitance of parallel plate condenser

$$C = \frac{\kappa A}{4\pi d}$$

Energy of condenser

$$W = \frac{q^2}{2C} = \frac{CV^2}{2} = \frac{qV}{2}$$

Energy of electric field per c.c.

$$W_1 = \frac{\kappa\varepsilon^2}{8\pi}$$

Force between magnetic poles

$$F = \frac{m_1 m_2}{r^2}$$

Force on conductor in a magnetic field

$F = lHI_a$

Magnetic field due to element of a circuit

$$\delta H = I_a \frac{\delta l}{r^2} \sin\theta$$

Magnetic moment of circuit

$$M = I_a a$$

Magnetic potential

$$V = I_a \Omega$$

Force on moving charge

$$F = Hqu$$

E.m.f. generated in conductor

$$E = lHu$$

Work done in increasing flux through a circuit

$$W = I_a \Delta\phi$$

Self-inductance defined by

$$e_1 = -L\frac{dI_a}{dt}$$

Mutual inductance defined by

$$e_2 = -M\frac{dI_a}{dt}$$

Energy associated with an inductance

$$W = \tfrac{1}{2}LI_a^2$$

M.m.f. round closed circuit

$$= 4\pi I_a$$

E.m.f. in circuit through which flux changes

$$e = -\frac{d\phi}{dt}$$

Component curls

$$(\text{Curl})_x = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}$$

$$(\text{Curl})_y = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}$$

$$(\text{Curl})_z = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

Relation between magnetic force, magnetic induction and intensity of magnetisation

$$B = H + 4\pi J$$

Maxwell's displacement current

$$I_a = \frac{1}{c}\frac{dN}{dt}$$

Field near long straight wire

$$H = \frac{2I_a}{r}$$

Internal inductance of wire per cm.

$L_i = \frac{1}{2}\mu$

Inductance of coaxial cable

$$L = 2 \times 10^{-9} \log_\epsilon \frac{r_2}{r_1} \text{ henries per cm. length}$$

Inductance of a pair of parallel wires

$$L = 4 \times 10^{-9} \log_\epsilon \frac{d}{r} \text{ henries per cm. length}$$

Capacitance of coaxial cable

$$C = \frac{1}{9 \times 10^{11}} \cdot \frac{\kappa}{2 \log_\epsilon \frac{r_2}{r_1}} \text{farads per cm. length}$$

Capacitance of a pair of parallel wires

approximately $C = \frac{1}{9 \times 10^{11}} \cdot \frac{\kappa}{4 \log_\epsilon \frac{d}{r}}$ farads per cm. length

Exact solution given by writing

$$\frac{\sqrt{d^2 - 4r^2} + (d - 2r)}{\sqrt{d^2 - 4r^2} - (d - 2r)} \text{ for } \frac{d}{r}$$

Dielectric stress in coaxial cable

$$\varepsilon = \frac{V}{r \log_\epsilon \frac{r_2}{r_1}}$$

Pull of circuit on piece of iron

$$P = \frac{1}{2} I_a^2 \frac{dL}{dx}$$

Velocity of slow moving electrons

$u = 5{\cdot}94 \times 10^7 \sqrt{V}$ cm./sec.

if V in volts

Electric deflection of cathode ray

Angular sensitivity in radians per volt $\frac{l_d}{2Vd}$

l_d = length of deflecting plates, d = distance between them, V = gun voltage.

Magnetic deflection of cathode ray

$$\theta = \frac{0{\cdot}297 H l_m}{\sqrt{V}} \text{ radians}$$

l_m = length of path in field H

APPENDIX III

A NOTE ON DIMENSIONS

The equations with which mathematics deals express the equality of two numbers. When mathematics is applied to scientific problems the equations express the numerical equality of certain measures abstracted from the entities which are the subjects of observation. The equations may be considered to go further, to express not only the numerical equality of the two terms but also an equality of kind. Except in the case when an equation may represent an identity such as

$$2 \text{ atoms of hydrogen} = 2 \text{ atoms of hydrogen}$$

(equations in which applied science has no interest), the terms of the two sides are not things or entities but measures of things or entities, or more strictly, measures of selected properties of things. Such measures are time, displacement, force, velocity, area, mass, colour, volume, density, entropy, etc. etc. Between these measures certain relations exist determined by necessity, or convention. It is only between the selected measures that any equality can exist. For instance, the equation

$$P = mf$$

may express the fact that a force P may be measured by the acceleration f which it can produce when acting upon a mass m. It does not mean that force is *equal* to mass multiplied by acceleration. Such a statement, though sometimes permissible for the sake of economy of language, is obviously absurd.

Certain measures, such for instance as length, mass and time (which suffice for mechanical problems), may be selected and other measures defined in terms of these *dimensions*. It is clear that any equation must present not only two quantities of numerical equality but also of dimensional equality (3 inches can never equal 3 seconds).

Difficulties may arise when the same entity is measured in different ways. For instance, an electric charge may be measured by the repulsive force which it exerts upon an equal charge ; or it may be measured by the intensity of the magnetic field which it produces when in motion. It has been seen that these measures

are dimensionally different, the ratio between them being a velocity. When it is remembered that what is measured is not the charge itself but some abstracted property of the charge, this difference of dimensions should present no difficulty. To some, however, this difference is repugnant and to obviate it various devices, none of which has gained general acceptance, are employed. It may be assumed that μ and κ are not dimensionless numbers but that their product in empty space has the dimensions of the square of a velocity. Various schools differ in their apportioning of the dimensions between μ and κ.

It will have been noted that little attention has been devoted to dimensions throughout the present book. It is felt that in recent years they have received an attention which might profitably have been directed to more important matters. Clear thinking is, however, essential and, when that is attained, difficulties of interpretation disappear. Uniform conventions would be useful but at present the prospect of their general acceptance seems remote. The author considers it unfortunate that, to some, B and H have different dimensions, but it is of little importance. Peas may be purchased by the pound or the peck. Pecks can never be expressed in pounds.

> "'Write that down', the King said to the jury, and the jury eagerly wrote down all three dates on their slates, and then added them up, and reduced the answer to shillings and pence'.
>
> (Lewis Carroll)

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INDEX

ADDITION, 1
Admittance, vector, 103
Aerial feeder, 280
Alternator, single phase, 198
Ammeter, moving iron, 88
Ampere, 54, 72
Amplitude, argument or phase, 4
Area, determination of, from polar diagram, 133
Argand diagram, 4
Argument, amplitude or phase, 4
Arrival curve, 285
Attenuation constant, 276, 278
Auxiliary equation, 153
Average value of function, 74

BALANCED 3-phase systems, 112, 114
Ballistic galvanometer, 188
Ber and bei functions, 246
Berg, 327
Bessel functions, Chaps. XVII, XVIII, 370
Beta functions, 136, 368

CAPACITANCE, 47
— of coaxial cable, 84
— of parallel plate condenser, 48
— — — wires, 84
— of sphere, 47
— of two concentric spheres, 48
Capacitative circuit, 148
— effect of moving coil, 209 et seq.
Cathode ray, electric deflection of, 93
— — magnetic deflection of, 94
Cauchy-Riemann differential equations, 330
C.G.S. units, 70
Characteristic impedance, 276
Charge of condenser, 149
— surface density of, 45
Circular current, magnetic field due to, 76, 78
— functions, 361, 369
Coaxial cable, capacitance of, 84
— — dielectric stress in, 87
— — inductance of, 83
— — optimum dimensions of, 87
— feeder, optimum dimensions of, 87
Commutative operators, 165, 220
Complementary function, 151, 173
Complex impedance, 102
— multiplier, 3
— number, Chap. I
— — expressed as exponential function, 7
— operator, 99
Condenser, 47
— current in, 148 et seq.
— discharge, 140, 149
— parallel plate, 48
— potential energy of, 49
Conductance, 103
Conductor whose breadth changes suddenly, 347
Conformal representations, 333
Conjugate functions, Chap. XXII
Constant of integration, 30, 140, 143
Corrected integral, 30
Coulomb, 72
Coupled circuits, 107, 195, 203
Critical damping, 179, 186, 187
Curl of a vector, 60, 64, 66
Current, displacement, 72
— electromagnetic unit of, 53

D OPERATOR, 154, Chap. XII
Damped vibration, 185
Damping, critical, 179, 186, 187
Decrement, logarithmic, 107, 178
Definite integrals, 30, 34
Degree of differential equation, 139
Dependent variable, 12
— — change of, 224
Derivative, 13
Derived function, 13
— systems of vectors, 116
Dielectric stress in coaxial cable, 87
Differential, 13, 28
— total, 120
Differential coefficient, 13
— — partial, 43
— — total, 122

Differential coefficients of standard forms, 364 et seq.
Differential equations, Chaps. X–XVI, XIX
— — Cauchy-Riemann, 330
— — degree of, 139
— — exact, 143, 216
— — homogeneous, 144, 214
— — linear, 145, Chap. XI, 172, Chaps. XIII–XV
— — order of, 139
— — ordinary, 139
— — partial, 139, Chap. XIX
— — simultaneous, Chap. XIV
— — solution in series, Chap. XVI
Differentiation, 12
— and integration, reciprocal relations, 15
— logarithmic, 22
— methods, Chap. III
— of function of a function, 21
— of inverse function, 22
— of product, 21
— of quotient, 21
— of standard forms, 20
— partial, 42, Chap. VIII
— results of, Chap. III
— successive, 22
— under an integral sign, 135
Dimensions, App. III
Discharge of condenser, 140, 149
Displacement current, 72
— electric, 46
Distortionless line, 276
Divergence of a vector, 44
Division, 2
Double integration, 131
Duhamel's superposition theorem, 314

EDDY current loss in iron, 252
Electric charges, force between, 40
— displacement, 46
— field, 40
— — energy of, 49
— flux density, 46
— potential, 40
— — due to point charge, 41
Electrical polarisation, 46
Electrodynamics, 53
— laws of, 70
Electromagnetic radiation, 268
Electromotive force, 62
— — generation of, in magnetic field, 57
Electrons, in electric field, 92
— in magnetic field, 94
Electrostatics, 40
Elliptic integral, 79
Energy, of condenser, 49
— of electric field, 49
Equations, differential—see Differential equations
Equipotential surface, replacement by conducting surface, 47
— surfaces, 41
Eulerian integrals, 135, 136
Euler's constant, 234
Exact differential equation, 143
— linear differential equations, 216
Expansion theorem, 310
Exponential function, 5, 362
— theorem, 6

FARADAY tube, 46
— — equality of charges at ends of, 46
Field, electric, 40
— — energy of, 49
— magnetic, 50
— vector, 59
Flat-topped wave, 372
Flux density, electric, 46
— distribution in air gap, 296
— linkages, 58
Force, between conductors, 80
— — electric charges, 40
— — magnetic poles, 50
— lines of, 40, 50
— on a conductor in magnetic field, 53
— on iron plunger in solenoid, 88
— on moving charge, 57
— tube of, 45
Fourier series, Chap. XX
Furnace, induction, 252

GALVANOMETER, ballistic, 191
— moving coil, 188
— vibration, 190
Gamma functions, 129, 136, 368
— — integral values of, 138
Gauss's theorem, 41
Gradient, 59

HALF range Fourier series, 289
Harmonic analysis, Chap. XX
— curves, 290
— quantities, Chap. VII
Heaviside, 160, 169, 278, 306

Heaviside's operational calculus, Chap. XXI.
— shifting theorem, 160, 169
Henry, 72
High-frequency current in conductor, 243, 258
— feeder, 87
Homogeneous differential equation, 144, 214
Howe, 248
$H(t)$, 308
Hyperbolic angles, 9
— functions, 8, 38, 171, 362 et seq., 369

IMAGINARY operator, 4
Impedance, characteristic, 87, 276
— complex, 102
— sending end, 279
— vector, 102
— — mechanical, 209
Indefinite integral, 30
Independent variable, 12
Indicial equation, 227
Inductance, 59
— mutual, 59
— of circuit, 82
— of coaxial cable, 83
— of parallel wires, 83
Induction furnace, 252
Inductive circuit, 146
Integral, corrected, 30
— definite, 30, 34
— elliptic, 79
— independent, 30
— line, 60
— uncorrected, 30
Integral forms of Bessel functions, 239
Integrals of standard forms, 366
Integrating factor, 143
Integration, 14
— by parts, 35
— double, 131
— general laws of, 31
— methods, Chap. IV, Chap. IX
— results, Chap. IV
— triple, 134
Integration constant, 30, 140
Intensity of magnetisation, 52
Interpretable operators, 166
Inverse operations upon unity, 173
— — upon zero, 173
Inverse operators, 166
— — expansion of, 167
Inversion, 103
— of a circle, 104
— of a straight line, 104
Iron, magnetisation of, 67

j (imaginary operator), 4
— arithmetical interpretation of, 5

KEMP'S method of harmonic analysis, 303
Ker and kei functions, 246

LAMINATION, penetration of flux into, 263
Laplace's equation, 45, 329
Lees, 347, 356
Legendre functions, 230
Legendre's equation, 229
Leibnitz's theorem, 24
Line, distortionless, 276
— equations, 272, 315, 324
— impedance, 278
— infinitely long, 276
— integral, 60
— — in the electric field, 61
— open, 279, 318
— shortcircuited, 279
— with small losses, 278, 318
Linear differential equations, 145, Chap. XI, 172, Chaps. XIII–XV
Lines of force, 40
Loading coils, 278
Logarithmic decrement, 107, 178
— expansion, 6
Loud speaker, 208

McLACHLAN, 229, 239, 254
Maclaurin's theorem, 25
Magnetic field, 50
— — produced by electric current, 53, 75–80
— flux, change of flux linkages with circuit, 58
— — penetration into lamination, 263
— moment, 50
— — of an electric circuit, 55
— potential, 50
— — due to electric current, 56
— shell, 51
Magnetisation, intensity of, 52, 70
— of iron, 67
Magnetism, 50
Magnetomotive force, 63
— — in air gap of 3-phase machine, 299
Mallett, 68
Maxima, 28, 123

Maxwell's displacement current, 72
Mechanical impedance, 209
— power factor, 209
Minima, 28, 123
Modified Bessel functions, 238
Modulus, 4
Moment, magnetic, 50, 55
Moving charge, force on, 57
— coil, capacitative effect of, 209 et seq.
Moving-coil galvanometer, 188
— speaker, 208
Moving iron ammeter, 88
Multiplication, 2, 3
— of complex numbers, 5
Murphy, R., 169
Mutual inductance, 59

NEGATIVE quantities, 2
Neumann's Bessel function, 233
Ninth harmonic, 304
Numbers real and complex, Chap. I

OPERATIONAL calculus, Chap. XXI
— forms, 369
— methods, Chap. XII
Operations upon unity, 173, 369
— — zero, 173, 370
Operator D, 154, Chap. XII
Operator, factorisation of, 155, 158
— imaginary, 4
Operator θ, 214
Operators, commutative, 165, 220
— interpretable, 166
— inverse, 166
Order of differential equations, 139
Oscillating power, 110, 114, 119
Oscillator, valve, 205
Oscillatory circuits, 177

p $(= \partial/\partial t)$, 167
Parallel plate condenser, 48
— wires, capacitance of, 85
— — inductance of, 83
Partial differential equations, 139, Chap. XIX
— differentiation, 43, Chap. VIII
— integration, 35
Particular integral, 151, 157, 159
Peaked wave form, 372
Penetration of flux into lamination, 263
Permeability, 67
Phase, amplitude or argument, 4
Pidduck, 70
Pinch effect, 81
Plane wave, 270
Point source, flow of current from, 356
Poisson's equation, 43
Polarisation, electric, 46
Polyphase systems, 110
Potential, 59
— electric, 40
— energy of condenser, 49
— magnetic, 50, 61
Power factor, mechanical, 209
— in an alternating current circuit, 110, 114, 119
Primitive, 140
Propagation constant, 276

QUARTER wave-length transformer, 280
Quasi-conjugate functions, 357

RADIATION, electromagnetic, 268
Radio feeder, 87
Ratio of transformation, 339
Rayleigh, 248
Reactance, 102
Real numbers, 5, Chap. I
Rectangle, magnetic field within, 80
Rectified current in thermionic valve 243
— sine wave, 299
Reduction formulæ, 127
Resistance, high frequency, 247 et seq., 258 et seq.
Resonance, 106, 186
Resonant circuit, 106
Root mean square value of function, 75
Rotating flux wave, 297
— vectors, 96 et seq.
Russell, 199, 248, 249, 254

SAW tooth wave, 372
Scalars, 2
Schwarz-Christoffel transformation, 343
Self-inductance, calculation of, 82
Sending end impedance, 279
Separation of variables, 144
Shifting theorem, 160, 169
Simple harmonic quantities, 97 et seq.

Simultaneous differential equations, Chap. XIV
Single-phase alternator, 198
Sinoidal function, average value of, 74
— — R.M.S. value of, 75
Solenoid, pull upon iron plunger, 88
Solenoidal vector, 45
Speaker, moving-coil, 208
Spherical wave, 270
Square, magnetic field within, 80
Standard forms, differential coefficients of, 364
— — integrals of, 366
Stream function, 329
Substitution, method of, 33
— — definite integrals by, 34
Subtraction, 2
Successive differentiation, 22
— — of a product, 24
— partial differentiation, 121
Superposition theorem, 314
Surface density of charge, 45
— zonal harmonics, 230
Susceptance, 103
Symmetrical components, 114
Synchronous spark transmitter, 183

TAYLOR'S theorem, 26
Telegraph equation, 283, 316
Telephone diaphragm, 212
Three-phase systems, 112
Time constant, 150
Total differential, 120
— — coefficient, 122
Transformation ratio, 339
— Schwartz-Christoffel, 343
Transformer, 195, 203
— quarter wave-length, 280
Transient, 182
Transmission along wires, 272, 318, 324
Transmitter, synchronous spark, 183
Trapezoidal wave, 296
Triple integration, 134
Tube, Faraday, 46
Tubes of force, 45

UNBALANCED 3-phase systems, 112, 119
Unit function, 306
Units, 70, 72
Unity, inverse operations upon, 174, 370

VALVE oscillator, 205
— rectified current in, 243
Variable, dependent, 12
— independent, 12
Vector, 2
— curl of, 60, 64
— divergence of, 44
— line integral of, 60
— rotating, 96
— solenoidal, 45
Vector admittance, 103
Vector fields, 59
— impedance, 102
— loci, 103
— operator, 99
— quantities, Chap. VII
Vectors, derived system of, 116
Velocity of transmission, 277
Vibration, damped, 185
— galvanometer, 190
Vinycomb, 57, 68
Volt, 72

WALKER, Miles, 347
Wave form, flat-topped, 372
— — peaked, 372
— — saw-toothed, 372
— — trapezoidal, 296
Wave, plane, 270
— spherical, 271
Wave-length constant, 276
Wave motion, vectorial representation of, 100
— number, 101
Weber's Bessel function, 233
Wire transmission, 272, 318, 324
Work done in changing flux linking circuit, 58
— — in increasing area of circuit, 91

ZERO, operations upon, 173, 370
Zonal harmonics, 230